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Characterizations of Slant Ruled Surfaces in the Euclidean 3-space

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Abstract. In this study, we give the relationships between the conical curvatures of ruled surfaces generated by the unit vectors of the ruling, central normal and central tangent of a ruled surface in the Euclidean 3-space E^3. We obtain differential equations characterizing slant ruled surfaces and we give the conditions for the surfaces generated by central normal and central tangent vectors to be slant ruled surfaces while the reference ruled surface is a slant ruled surface.

Keywords: Slant ruled surface; Frenet frame; Conical curvature; Differential equation.

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1. Introduction

In differential geometry of curves and surfaces, special curves and surfaces have an important role and more applications. Generally, special curves are such curves whose curvatures satisfy some special conditions. One of the well-known of such curves is the helix curve in the Euclidean 3-space E^3 which is defined by the property that the tangent line of the curve makes a constant angle with a fixed straight line called the axis

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of the general helix [2]. Therefore, a general helix can be equivalently defined as one whose tangent indicatrix is a planar curve.

Recently, another special curve similar to helix, called slant helix, has been defined by Izumiya and Takeuchi [3]. They defined a slant helix such as the normal lines of the curve make a constant angle with a fixed direction and they have given a characterization for slant helix in the Euclidean 3-space E^3. Moreover, slant helices have been studied by some mathematicians and new types of these curves have been introduced. Kula and Yaylı have investigated spherical images, the tangent indicatrix and the binormal indicatrix of a slant helix and they have obtained that the spherical images of a slant helix are helices lying on unit sphere [5]. Kula and et al have introduced some new results characterizing slant helices in E^3 [6]. Ali has studied the position vectors of slant helices in E^3 [1]. Monterde has shown that for a curve with constant curvature and non-constant torsion, the principal normal vector of the curve makes a constant angle with a fixed constant direction, i.e., the curve is a slant helix [7]. Later, Önder and et al have defined a new type of slant helix called B_2-slant helix in the Euclidean 4-space E^4 by saying that the second binormal vector of a space curve makes a constant angle with a fixed direction in E^4 [10].

In the case of surfaces, ruled surfaces are a kind of special surfaces which are generated by a continuous movement of a line along a curve. Önder has considered the notion of "slant helix" for ruled surfaces and defined slant ruled surfaces in E^3 by the property that the vectors of the Frenet frame of the surface make constant angles with fixed directions and he has given characterizations for a ruled surface to be a slant ruled surface [9]. He has also shown that the striction curves of developable slant ruled surfaces are helices or slant helices. Later, Önder and Kaya have defined Darboux slant ruled surfaces in E^3 such as the Darboux vector of the ruled surface makes a constant angle with a fixed direction and they have given characterizations for a ruled surface to be a Darboux slant ruled surface [8].

In this work, we give new characterizations for slant ruled surfaces in E^3. These characterizations are given by differential equations of Frenet vectors.

2. Ruled Surfaces in the Euclidean 3-space E^3

In this section, we give a brief summary of the geometry of ruled surfaces in E^3. A more detailed information can be found in [4].

A ruled surface S is a special surface generated by a continuous moving of a line along a curve and has the parametrization

$$\vec{r}(u,v) = \vec{f}(u) + v\,\vec{q}(u), \tag{2.1}$$

where$\vec{f} = \vec{f}(u)$ is a regular curve in E^3 defined on an open interval $I \subset \mathbb{R}$ and $\vec{q} = \vec{q}(u)$ is a unit direction vector of an oriented line in E^3. The curve $\vec{f} = \vec{f}(u)$ is called base curve or generating curve of the surface and various positions of generating lines $\vec{q} = \vec{q}(u)$ are called rulings. In particular, if the direction of $\vec{q}$ is constant, then the ruled surface is said to be cylindrical, and non-cylindrical otherwise.

Let $\vec{m}$ be unit normal vector of ruled surface S. Then we have

$$\vec{m} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} = \frac{(\dot{\vec{f}} + v\dot{\vec{q}}) \times \vec{q}}{\sqrt{\left\langle \dot{\vec{f}} + v\dot{\vec{q}},\ \dot{\vec{f}} + v\dot{\vec{q}} \right\rangle - \left\langle \dot{\vec{f}}, \vec{q} \right\rangle^2}}. \tag{2.2}$$

If v infinitely decreases, then along a ruling $u = u_1$, the unit normal $\vec{m}$ approaches a limiting direction. This direction is called the asymptotic normal (central tangent) direction and from (2.2) defined by

$$\vec{a} = \lim_{v \to \pm\infty} \vec{m}(u_1, v) = \frac{\vec{q} \times \dot{\vec{q}}}{\left\|\dot{\vec{q}}\right\|}.$$

The point at which the unit normal of S is perpendicular to $\vec{a}$ is called the striction point (or central point) C and the set of striction points of all rulings is called striction curve of the surface.

The vector $\vec{h}$ defined by $\vec{h} = \vec{a} \times \vec{q}$ is called central normal vector. Then the orthonormal system $\left\{C;\ \vec{q}, \vec{h}, \vec{a}\right\}$ is called Frenet frame of the ruled surface S where C is the central point and $\vec{q},\ \vec{h},\ \vec{a}$ are unit vectors of ruling, central normal and central tangent, respectively.

The set of all bound vectors $\vec{q}(u)$ at origin O constitutes a cone which is called directing cone of the ruled surface S. The end points of unit vectors $\vec{q}(u)$ drive a spherical curve k_q on the unit sphere S_1^2 and this curve is called spherical image of ruled surface S, whose arc length is denoted by s_q.

For the Frenet formulae of the ruled surface S and of its directing cone with respect to the arc length s_q we have

$$\begin{bmatrix} d\vec{q}/ds_q \\ d\vec{h}/ds_q \\ d\vec{a}/ds_q \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \kappa_q \\ 0 & -\kappa_q & 0 \end{bmatrix} \begin{bmatrix} \vec{q} \\ \vec{h} \\ \vec{a} \end{bmatrix}, \tag{2.3}$$

where $\kappa_q(s_q) = \|d\vec{a}/ds_q\|$ is called the conical curvature of the directing cone (For details [4]).

Definition 2.1. ([10]) Let S_q be a ruled surface in E^3 given by the parametrization

$$\vec{r}(s, v) = \vec{c}(s) + v\,\vec{q}(s), \quad \|\vec{q}(s)\| = 1,$$

where $\vec{c}(s)$ is striction curve of S and s is arc length parameter of $\vec{c}(s)$. Let the Frenet frame of S be $\left\{\vec{q},\vec{h},\vec{a}\right\}$. Then S is called a q-slant (h-slant or a-slant, respectively) ruled surface if the ruling (the vector $\vec{h}$ or the vector $\vec{a}$, respectively) makes a constant angle with a fixed non-zero direction $\vec{u}$ in the space, i.e.,

$$\langle\vec{q},\vec{u}\rangle=\cos\theta=constant;\ \ \theta\neq\frac{\pi}{2}, \tag{2.4}$$

($\left\langle\vec{h},\vec{u}\right\rangle=\cos\theta=constant;\ \ \theta\neq\frac{\pi}{2}$ or $\langle\vec{a},\vec{u}\rangle=\cos\theta=constant;\ \ \theta\neq\frac{\pi}{2}$, respectively).

Theorem 2.2. *([8]) Let S_q be a ruled surface in E^3 with Frenet frame $\left\{\vec{q},\vec{h},\vec{a}\right\}$ and conical curvature $\kappa_q\neq 0$. Then S_q is an h-slant ruled surface if and only if the function*

$$\frac{\kappa_q'}{\left(1+\kappa_q^2\right)^{3/2}}, \tag{2.5}$$

is constant.

Now, we will give the following theorem characterizing q-slant ruled surface in E^3.

Theorem 2.3. *Let S_q be a ruled surface in E^3 with Frenet frame $\left\{\vec{q},\vec{h},\vec{a}\right\}$ and conical curvature $\kappa_q\neq 0$. Then S_q is a q-slant ruled surface if and only if the function κ_q is constant.*

Proof. Let S_q be a q-slant ruled surface in E^3. Then denoting by $\vec{u}$ the unit vector of fixed direction and by θ the angle between $\vec{q}$ and $\vec{u}$, the following equality is satisfied

$$\langle\vec{q},\vec{u}\rangle=\cos\theta=constant. \tag{2.6}$$

By taking the derivative of (2.6) with respect to s_q gives $\left\langle\vec{h},\vec{u}\right\rangle=0$. Therefore, the vector $\vec{u}$ lies on the plane spanned by the vectors $\vec{q}$ and $\vec{a}$, i.e.,

$$\vec{u}=(\cos\theta)\vec{q}+(\sin\theta)\vec{a}. \tag{2.7}$$

By differentiating (2.7) with respect to s_q and considering that the direction of $\vec{u}$ is constant, it follows

$$0=(\cos\theta-\kappa_q\sin\theta)\vec{h},$$

and since $\vec{h}\neq\vec{0}$, we have $\kappa_q=\cot\theta$ is constant.

Conversely, let $\kappa_q=\cot\theta$ be constant. We define,

$$\vec{u}=(\cos\theta)\vec{q}+(\sin\theta)\vec{a}.$$

Differentiating the last equation and using $\kappa_q = \cot\theta$ is constant we get $\vec{u}' = 0$, i.e., $\vec{u}$ is a constant vector. On the other hand, $\langle \vec{q}, \vec{u} \rangle = \cos\theta =$ *constant*. Then we conclude that S_q is a q-slant ruled surface. □

3. Frenet Formulae of the Ruled Surfaces Generated by the Central Normal Vector and Central Tangent Vector

In this section, we give the Frenet formulae of ruled surfaces generated by central normal vector $\vec{h}$ and central tangent vector $\vec{a}$ of the Frenet frame $\left\{ \vec{q}, \vec{h}, \vec{a} \right\}$ of a regular ruled surface S_q. We show the ruled surfaces generated by $\vec{h}$ and $\vec{a}$ by S_h and S_a, respectively; and their Frenet formulae and conical curvatures by $\left\{ \vec{q}_h, \vec{h}_h, \vec{a}_h \right\}$, κ_h and $\left\{ \vec{q}_a, \vec{h}_a, \vec{a}_a \right\}$, κ_a, respectively.

First, we give the following theorem giving relations between the curvatures and the Frenet formulae of surfaces S_q and S_h generated by $\vec{q}$ and $\vec{h}$, respectively.

Theorem 3.1. *Let S_q be a ruled surface in E^3 with the Frenet frame $\left\{ \vec{q}, \vec{h}, \vec{a} \right\}$ and with non-zero conical curvature κ_q. Then the relationship between the conical curvatures of the surfaces S_q and S_h is given by*

$$\kappa_h = \frac{\kappa_q'}{\left(1+\kappa_q^2\right)^{3/2}}.$$

Then the Frenet formulae of S_h are

$$\begin{bmatrix} d\vec{q}_h/ds_h \\ d\vec{h}_h/ds_h \\ d\vec{a}_h/ds_h \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \kappa_h \\ 0 & -\kappa_h & 0 \end{bmatrix} \begin{bmatrix} \vec{q}_h \\ \vec{h}_h \\ \vec{a}_h \end{bmatrix}. \tag{3.1}$$

Proof. For the parametrization of S_h, we have

$$\vec{r}_h(s,v) = \vec{c}(s) + v\,\vec{h}(s), \quad \left\| \vec{h}(s) \right\| = 1,$$

where $\vec{c}(s)$ is striction curve of S_q and s is arc length parameter of $\vec{c}(s)$. Since the Frenet frame of S_h is given by $\left\{ \vec{q}_h, \vec{h}_h, \vec{a}_h \right\}$, we can write

$$\vec{q}_h = \vec{h},$$

and if we use the Frenet formulae given by (2.3), we get the central normal vector of S_h as,

$$\vec{h}_h = \frac{d\vec{q}_h}{ds_q}\frac{ds_q}{ds_h} = (-\vec{q} + \kappa_q \vec{a})\frac{ds_q}{ds_h}, \tag{3.2}$$

where s_h is the arc length parameter of the spherical curve drawn by $\vec{h}$. Since $\vec{h}_h$ is a unit vector, from (3.2) we have

$$\frac{ds_q}{ds_h} = \frac{1}{\sqrt{1+\kappa_q^2}}.$$

Then (3.2) becomes

$$\vec{h}_h = \frac{1}{\sqrt{1+\kappa_q^2}}(-\vec{q} + \kappa_q \vec{a}). \tag{3.3}$$

Since $\vec{q}_h = \vec{h}$, from (3.3), the central tangent vector is

$$\vec{a}_h = \vec{q}_h \times \vec{h}_h = \frac{1}{\sqrt{1+\kappa_q^2}}(\vec{a} + \kappa_q \vec{q}),$$

and the conical curvature of S_h is

$$\kappa_h = \left\| \frac{d\vec{a}_h}{ds_h} \right\| = \left\| \frac{d\vec{a}_h}{ds_q}\frac{ds_q}{ds_h} \right\| = \frac{\kappa_q'}{\left(1+\kappa_q^2\right)^{3/2}}, \tag{3.4}$$

where $\kappa_q' = \frac{d\kappa_q}{ds_q}$ and we have,

$$\begin{bmatrix} d\vec{q}_h/ds_h \\ d\vec{h}_h/ds_h \\ d\vec{a}_h/ds_h \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \kappa_h \\ 0 & -\kappa_h & 0 \end{bmatrix} \begin{bmatrix} \vec{q}_h \\ \vec{h}_h \\ \vec{a}_h \end{bmatrix}.$$

where $\kappa_h = \frac{\kappa_q'}{\left(1+\kappa_q^2\right)^{3/2}}$. □

From Theorem 3.1 we can give the following theorem:

Theorem 3.2. *The surface S_q is an h-slant ruled surface if and only if the surface S_h is a q-slant ruled surface.*

Moreover, the relations between the curvatures and the Frenet formulae of surfaces S_q and S_a can be given as follows:

Theorem 3.3. *Let S_q be a ruled surface in E^3 with the Frenet frame $\left\{\vec{q}, \vec{h}, \vec{a}\right\}$ and with non-zero conical curvature κ_q. Then the relationship between the conical curvatures of the surfaces S_q and S_a is given by $\kappa_a = \frac{1}{\kappa_q}$. Then the Frenet formulae of S_a are*

$$\begin{bmatrix} d\vec{q}_a/ds_a \\ d\vec{h}_a/ds_a \\ d\vec{a}_a/ds_a \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \kappa_a \\ 0 & -\kappa_a & 0 \end{bmatrix} \begin{bmatrix} \vec{q}_a \\ \vec{h}_a \\ \vec{a}_a \end{bmatrix}. \tag{3.5}$$

Proof. For the parametrization of the surface S_a we have

$$\vec{r}_a(s,v) = \vec{c}(s) + v\,\vec{a}(s), \quad \|\vec{a}(s)\| = 1,$$

where $\vec{c}(s)$ is striction curve of S_q and s is arc length parameter of $\vec{c}(s)$. Since the Frenet frame of S_a is given by $\left\{\vec{q}_a, \vec{h}_a, \vec{a}_a\right\}$, we can write,

$$\vec{q}_a = \vec{a},$$

and use the Frenet formulae given by (2.3), we obtain the central normal vector $\vec{h}_a$ as follows,

$$\vec{h}_a = \frac{d\vec{q}_a}{ds_q}\frac{ds_q}{ds_a} = -\kappa_q \vec{h}_q \frac{ds_q}{ds_a}, \tag{3.6}$$

where s_a is the arc length parameter of the spherical curve drawn by $\vec{a}$. Since $\vec{h}_a$ is a unit vector, from (3.6) we have followings,

$$\frac{ds_q}{ds_a} = \frac{1}{\kappa_q}, \vec{h}_a = -\vec{h}.$$

Then the central tangent vector of the surface S_a is

$$\vec{a}_a = \vec{q}_a \times \vec{h}_a = \vec{q},$$

and the conical curvature of S_a is

$$\kappa_a = \left\|\frac{d\vec{a}_a}{ds_a}\right\| = \left\|\frac{d\vec{a}_a}{ds_q}\frac{ds_q}{ds_a}\right\| = \frac{1}{\kappa_q}.$$

Therefore we have

$$\begin{bmatrix} d\vec{q}_a/ds_a \\ d\vec{h}_a/ds_a \\ d\vec{a}_a/ds_a \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \kappa_a \\ 0 & -\kappa_a & 0 \end{bmatrix} \begin{bmatrix} \vec{q}_a \\ \vec{h}_a \\ \vec{a}_a \end{bmatrix},$$

where $\kappa_a = \frac{1}{\kappa_q}$. □

From Theorem 3.3, we have the following theorem:

Theorem 3.4. *The surface S_q is a q-slant ruled surface if and only if the surface S_a is a q-slant ruled surface.*

4. Differential Equation Characterizations for Slant Ruled Surfaces in E^3

In this section, we give the differential equations characterizing slant ruled surfaces. We begin with the following theorem characterizing q-slant ruled surfaces.

Theorem 4.1. *Let S_q be a ruled surface in E^3 with the Frenet frame $\left\{\vec{q},\vec{h},\vec{a}\right\}$ and non-zero conical curvature κ_q. Then, S_q is a q-slant ruled surface if and only if the vector $\vec{q}$ satisfies the following differential equation,*

$$\vec{q}\,''' + (1+\kappa_q^2)\vec{q}\,' = 0, \tag{4.1}$$

where $\vec{q}\,' = d\vec{q}/ds_q$, $\vec{q}\,''' = d^3\vec{q}/ds_q^3$.

Proof. Let S_q be a q-slant ruled surface in E^3. From the Frenet formulae given in (2.3) we have,

$$\vec{q}\,' = \vec{h}.$$

If we take the derivative of the last equation twice with respect to s_q, we obtain

$$\vec{q}\,''' = -\vec{h} + \kappa_q'\vec{a} - \kappa_q^2\vec{h}. \tag{4.2}$$

Since S_q is a q-slant ruled surface from Theorem 2.3, $\kappa_q = \text{constant}$, i.e., $\kappa_q' = 0$. Hence, from (4.2) it follows

$$\vec{q}\,''' + (1+\kappa_q^2)\vec{q}\,' = 0,$$

which is desired.

Conversely, let us assume that the equation (4.1) holds. From the Frenet formulae in (2.3) we have,

$$\vec{h}\,' = -\vec{q} + \kappa_q\vec{a}. \tag{4.3}$$

Since $\kappa_q \neq 0$, from (4.3) we can write

$$\vec{a} = \frac{1}{\kappa_q}(\vec{h}\,' + \vec{q}), \tag{4.4}$$

and by taking derivative of (4.4) we get

$$\vec{a}\,' = \frac{1}{\kappa_q}\left(\vec{h}\,'' + \vec{q}\,'\right) - \frac{\kappa_q'}{\kappa_q^2}\left(\vec{h}\,' + \vec{q}\right).$$

By using the Frenet formulae given in (2.3) and equation (4.1), from the last equation we obtain

$$\frac{\kappa_q'}{\kappa_q}\vec{a} = 0,$$

where $\kappa_q' = 0$ which gives us that $\kappa_q = \text{constant}$. Then, from Theorem 2.3, S_q is a q-slant ruled surface. □

Following two theorems give another differential equations characterizing again q-slant ruled surfaces.

Theorem 4.2. *Let S_q be a ruled surface in E^3 with the Frenet frame $\left\{\vec{q}, \vec{h}, \vec{a}\right\}$ and non-zero conical curvature κ_q. Then, S_q is a q-slant ruled surface if and only if the vector $\vec{h}$ satisfies the following differential equation*

$$\vec{h}'' + (1 + \kappa_q^2)\vec{h} = 0. \tag{4.5}$$

Proof. Let S_q be a q-slant ruled surface in E^3. From the Frenet formulae in (2.3) we have

$$\vec{h}' = -\vec{q} + \kappa_q \vec{a}.$$

By taking the derivative of the last equation with respect to s_q, we obtain

$$\vec{h}'' = -\vec{h} + \kappa_q' \vec{a} - \kappa_q^2 \vec{h}. \tag{4.6}$$

Since S_q is a q-slant ruled surface, from Theorem 2.3, κ_q = constant. Hence, from (4.6) it follows,

$$\vec{h}'' + (1 + \kappa_q^2)\vec{h} = 0.$$

Conversely, let the equation (4.5) holds. From (2.3) we have

$$\vec{h}' = -\vec{q} + \kappa_q \vec{a}.$$

By taking derivative of the last equation with respect to s_q, we get

$$\vec{q}' = -\vec{h}'' + \kappa_q' \vec{a} - \kappa_q^2 \vec{h},$$

and finally, by using (2.3) and (4.5) we obtain

$$\vec{h} = \vec{h} + \kappa_q' \vec{a},$$

which gives us $\kappa_q' = 0$, and so, κ_q = constant. Then Theorem 2.2 gives that S_q is a q-slant ruled surface in E^3. □

Theorem 4.3. *Let S_q be a ruled surface in E^3 with the Frenet frame $\left\{\vec{q}, \vec{h}, \vec{a}\right\}$ and with non-zero conical curvature κ_q. Then, S_q is a q-slant ruled surface if and only if the central tangent vector $\vec{a}$ satisfies the following differential equation*

$$\vec{a}''' + (1 + \kappa_q^2)\vec{a}' = 0. \tag{4.7}$$

Proof. Since S_q is a q-slant ruled surface, κ_q = constant, i.e., $\kappa_q' = 0$. Then, from (2.3) we have

$$\vec{a}' = -\kappa_q \vec{h}. \tag{4.8}$$

If we take derivative of the equation (4.8) and use the Frenet formulae given by (2.3), it follows

$$\vec{a}''' + (1 + \kappa_q^2)\vec{a}' = 0.$$

Conversely, let (4.7) holds. From the Frenet formulae in (2.3) we have

$$\vec{a}\,' = -\kappa_q \vec{h}.$$

If we differentiate the last equation twice, we get

$$2\kappa_q^{'}\vec{q} - \kappa_q^{''}\vec{h} - 3\kappa_q\kappa_q^{'}\vec{a} = 0. \tag{4.9}$$

Since the Frenet frame is linearly independent, from (4.9) we obtain following system

$$2\kappa_q^{'} = 0, \;\; \kappa_q^{''} = 0, \;\; 3\kappa_q\kappa_q^{'} = 0,$$

which gives us that κ_q = constant. Therefore, S_q is a q-slant ruled surface. □

In the following theorems, we give the differential equation characterizations for the surface S_q by means of the Frenet vectors of ruled surfaces S_h and S_a.

Theorem 4.4. *Let S_q be a ruled surface in E^3 with the Frenet frame $\left\{\vec{q}, \vec{h}, \vec{a}\right\}$ and with non-zero conical curvature κ_q. Then, S_q is an h-slant ruled surface if and only if the ruling vector $\vec{q}_h$ of the ruled surface S_h satisfies the following equation*

$$\frac{d^3\vec{q}_h}{ds_h^3} + (1+\kappa_h^2)\frac{d\vec{q}_h}{ds_h} = 0 \tag{4.10}$$

Proof. Let S_q be an h-slant ruled surface. Then from Theorem 2.2 we have that $\frac{\kappa_q'}{\left(1+\kappa_q^2\right)^{3/2}}$ is constant. From (3.1) we have

$$\frac{d\vec{q}_h}{ds_h} = \vec{h}_h. \tag{4.11}$$

Differentiating (4.11) twice and using that $\kappa_h = \frac{\kappa_q'}{\left(1+\kappa_q^2\right)^{3/2}}$ is constant, we get

$$\frac{d^3\vec{q}_h}{ds_h^3} + (1+\kappa_h^2)\frac{d\vec{q}_h}{ds_h} = 0,$$

which is desired.

Conversely, let us assume that the equation (4.10) holds. From (3.1) we have

$$\frac{d\vec{q}_h}{ds_h} = \vec{h}_h.$$

By taking derivative of the last equation with respect to s_h and using (4.10) it follows $\kappa_h'\vec{a}_h = 0$ which gives that $\kappa_h = \frac{\kappa_q'}{\left(1+\kappa_q^2\right)^{3/2}}$ is constant, and from Theorem 2.2 we have that S_q is an h-slant ruled surface. □

Theorem 4.5. *Let S_q be a ruled surface in E^3 with the Frenet frame $\left\{\vec{q}, \vec{h}, \vec{a}\right\}$ and with non-zero conical curvature κ_q. Then, S_q is an h-slant ruled surface if and only if the central normal vector $\vec{h}_h$ of the ruled surface S_h satisfies the equation*

$$\frac{d^2 h_h}{ds_h^2} + (1+\kappa_h^2)\vec{h}_h = 0 \tag{4.12}$$

where κ_h is the conical curvature of the surface S_h.

Proof. Let S_q be an h-slant ruled surface. From (2.6) we have

$$\frac{dh_h}{ds_h} = -\vec{q}_h + \kappa_h \vec{a}_h.$$

If we take derivative of the last equation we obtain

$$\frac{d^2 h_h}{ds_h^2} = -\vec{h}_h + \frac{d\kappa_h}{ds_h}\vec{a}_h - \kappa_h^2 \vec{h}_h. \tag{4.13}$$

Since S is an h-slant ruled surface then, from Theorem 2.2 and Theorem 3.1.

$$\kappa_h = \frac{\kappa_q'}{\left(1+\kappa_q^2\right)^{3/2}},$$

is constant. Then from (4.13) we have

$$\frac{d^2 h_h}{ds_h^2} + (1+\kappa_h^2)\vec{h}_h = 0. \tag{4.14}$$

Conversely, let us assume that (4.12) holds. From (3.1) we have that

$$\frac{d^2 h_h}{ds_h^2} = -\vec{q}\,'_h + \frac{d\kappa_h}{ds_h}\vec{a}_h - \kappa_h^2 \vec{h}_h. \tag{4.15}$$

If we substitute (4.12) in (4.15), we obtain

$$\frac{d\kappa_h}{ds_h}\vec{a}_h = 0,$$

which means that $\kappa_h =$ constant. Then from Theorem 3.1 and Theorem 2.2, S_q is an h-slant ruled surface in E^3. □

From Theorem 2.3 and Theorem 4.5 we have the following corollary:

Corollary 4.6. *S_h is a q-slant ruled surfaces if and only if $\frac{d^2 h_h}{ds_h^2} + (1+\kappa_h^2)\vec{h}_h = 0$, holds where κ_h is the conical curvature of the surface S_h.*

Theorem 4.7. *Let S_q be a ruled surface in E^3 with the Frenet frame $\left\{\vec{q}, \vec{h}, \vec{a}\right\}$ and with non-zero conical curvature κ_q. Then, S_q is an h-slant ruled surface if and only if the central tangent vector $\vec{a}_h$ of the ruled surface S_h satisfies following differential equation*

$$\frac{d^3\vec{a}_h}{ds_h^3} + (1+\kappa_h^2)\frac{d\vec{a}_h}{ds_h} = 0. \tag{4.16}$$

Proof. Let S_q be an h-slant ruled surface in E^3. From the Frenet formulae in (3.1) we have

$$\frac{d\vec{a}_h}{ds_h} = -\kappa_h\vec{h}_h, \tag{4.17}$$

By taking the derivative of (4.17) we get

$$\frac{d^2\vec{a}_h}{ds_h^2} = -\frac{d\kappa_h}{ds_h}\vec{h}_h - \kappa_h\left(-\vec{q}_h + \kappa_h\vec{a}_h\right) \tag{4.18}$$

Since S_q is an h-slant ruled surface in E^3, then, κ_h = constant, i.e., $\kappa_h' = 0$. Hence, (4.18) turns into

$$\frac{d^2\vec{a}_h}{ds_h^2} = -\kappa_h\left(-\vec{q}_h + \kappa_h\vec{a}_h\right). \tag{4.19}$$

By taking derivative of (4.19) and using $\kappa_h' = 0$, we obtain

$$\frac{d^3\vec{a}_h}{ds_h^3} + (1+\kappa_h^2)\frac{d\vec{a}_h}{ds_h} = 0,$$

which is desired.

Conversely, let us assume that (4.16) holds. From (3.1) we have

$$\frac{d\vec{a}_h}{ds_h} = -\kappa_h\vec{h}_h. \tag{4.20}$$

By differentiating (4.20) twice and using (4.16), we get

$$\kappa_h\vec{h}_h = 2\frac{d\kappa_h}{ds_h}\vec{q}_h + \left(\kappa_h - \frac{d^2\kappa_h}{ds_h^2}\right)\vec{h}_h - 3\kappa_h\frac{d\kappa_h}{ds_h}\vec{a}_h. \tag{4.21}$$

Since the Frenet frame is linearly independent, from (4.21) we obtain the following system:

$$\frac{d\kappa_h}{ds_h} = 0, \quad \kappa_h\frac{d\kappa_h}{ds_h} = 0, \quad \frac{d^2\kappa_h}{ds_h^2} = 0,$$

which gives us that κ_h = constant. Then from Theorem 2.3 and Theorem 3.2, we have that S_q is an h-slant ruled surface in E^3. □

From Theorem 2.2 and Theorem 4.7 we have the following corollary:

Corollary 4.8. *S_h is a q-slant ruled surfaces if and only if $\frac{d^3\vec{a}_h}{ds_h^3}+(1+\kappa_h^2)\frac{d\vec{a}_h}{ds_h}=0$, holds where κ_h is the conical curvature of the surface S_h.*

Another characterizations for q-slant ruled surfaces can be given as follows.

Theorem 4.9. *Let S_q be a ruled surface in E^3 with the Frenet frame $\left\{\vec{q},\vec{h},\vec{a}\right\}$ and with non-zero conical curvature κ_q. Then, S_q is a q-slant ruled surface if and only if the ruling vector $\vec{q}_a$ of the ruled surface S_a satisfies the following differential equation*

$$\frac{d^3\vec{q}_a}{ds_a^3}+(1+\kappa_a^2)\frac{d\vec{q}_a}{ds_a}=0. \tag{4.22}$$

Proof. Let S_q be an q-slant ruled surface in E^3. From the Frenet formulae in (3.5) we have

$$\frac{d\vec{q}_a}{ds_a}=\vec{h}_a.$$

If we take derivative of the last equation twice with respect to s_a, we get

$$\frac{d^3\vec{q}_a}{ds_a^3}=-\frac{d\vec{q}_a}{ds_a}+\frac{d\kappa_a}{ds_a}\vec{a}_a-\kappa_a^2\frac{d\vec{q}_a}{ds_a}. \tag{4.23}$$

Since S_q is a q-slant ruled surface in E^3, then from Theorem 2.3 and Theorem 3.3, κ_a is constant, i.e., $\frac{d\kappa_a}{ds_a}=0$. Then from (4.23) it follows

$$\frac{d^3\vec{q}_a}{ds_a^3}+(1+\kappa_a^2)\frac{d\vec{q}_a}{ds_a}=0,$$

which is desired.

Conversely, let (4.22) holds. From (3.5) we have

$$\frac{d\vec{h}_a}{ds_a}=-\vec{q}_a+\kappa_a\vec{a}_a. \tag{4.24}$$

By taking derivative of (4.24) we get

$$\frac{d^2\vec{h}_a}{ds_a^2}=\frac{d^3\vec{q}_a}{ds_a^3}=-\frac{d\vec{q}_a}{ds_a}-\kappa_a^2h_a+\frac{d\kappa_a}{ds_a}\vec{a}_a, \tag{4.25}$$

and using (4.22) in (4.25), we obtain that $\kappa_a=$ constant, which means that $\kappa_q=$ constant and so S_q is an q-slant ruled surface in E^3. □

Theorem 4.10. *Let S_q be a ruled surface in E^3 with the Frenet frame $\left\{\vec{q},\vec{h},\vec{a}\right\}$ and with non-zero conical curvature κ_q. Then, S_q is a q-slant* *ruled surface if and only if the central normal vector $\vec{h}_a$ of the ruled surface S_a satisfies the following differential equation*

$$\frac{d^2\vec{h}_a}{ds_a^2} + (1+\kappa_a^2)\vec{h}_a = 0. \tag{4.26}$$

Proof. Let S_q be a q-slant ruled surface in E^3. From the Frenet formulae in (3.5) we have,

$$\frac{d\vec{h}_a}{ds_a} = -\vec{q}_a + \kappa_a \vec{a}_a.$$

By taking derivative of the last equation we get

$$\frac{d^2\vec{h}_a}{ds_a^2} = -\frac{d\vec{q}_a}{ds_{a\,a}} + \frac{d\kappa_a}{ds_a}\vec{a}_a - \kappa_a^2 h_a. \tag{4.27}$$

Since S_q is a q-slant ruled surface inE^3, from Theorem 2.3, $\kappa_a =$ constant. Then from (4.27) we have

$$\frac{d^2\vec{h}_a}{ds_a^2} + (1+\kappa_a^2)\vec{h}_a = 0.$$

Conversely, let the equation (4.26) holds. From (3.5) we have

$$\vec{q}_a = \kappa_a \vec{a}_a - \frac{d\vec{h}_a}{ds_a}. \tag{4.28}$$

By taking derivative of (4.28) and using (3.5) and (4.26) we conclude that $\kappa_a =$ constantwhich means that S_q is a q-slant ruled surface in E^3. □

Theorem 4.11. *Let S_q be a ruled surface in E^3 with the Frenet frame $\left\{\vec{q}, \vec{h}, \vec{a}\right\}$ and with non-zero conical curvature κ_q. Then, S_q is a q-slant ruled surface if and only if the central tangent vector $\vec{a}_a$ of the ruled surface S_a satisfies following differential equation*

$$\frac{d^3\vec{a}_a}{ds_a^3} + (1+\kappa_a^2)\frac{d\vec{a}_a}{ds_a} = 0. \tag{4.29}$$

Proof. Let S_q be a q-slant ruled surface in E^3. From the Frenet formulae in (3.5) we have

$$\frac{d\vec{a}_a}{ds_a} = -\kappa_a \vec{h}_a.$$

By taking derivative of the last equation we get

$$\frac{d^2\vec{a}_a}{ds_a^2} = -\frac{d\kappa_a}{ds_a}\vec{h}_a + \kappa_a \vec{q}_a - \kappa_a^2 \vec{a}_a. \tag{4.30}$$

Since S_q is a q-slant ruled surface in E^3, $\kappa_a =$ constant and from (4.30) it follows,

$$\frac{d^2\vec{a}_a}{ds_a^2} = \kappa_a \vec{q}_a - \kappa_a^2 \vec{a}_a. \tag{4.31}$$

By taking derivative of (4.31) again and using $\frac{d\kappa_a}{ds_a} = 0$ we obtain

$$\frac{d^3\vec{a}_a}{ds_a^3} + (1+\kappa_a^2)\frac{d\vec{a}_a}{ds_a} = 0,$$

which is desired.

Conversely, let us assume that (4.29) holds. From (3.5) we have

$$\frac{d\vec{a}_a}{ds_a} = -\kappa_a \vec{h}_a. \tag{4.32}$$

By taking derivative of (4.32) twice and using (4.29) we get

$$\kappa_a \vec{h}_a = 2\frac{d\kappa_a}{ds_a}\vec{q}_a + \left(\kappa_a - \frac{d^2\kappa_a}{ds_a^2}\right)\vec{h}_a - 3\kappa_a \frac{d\kappa_a}{ds_a}\vec{a}_a. \tag{4.33}$$

Since the Frenet frame is linearly independent, from (4.33) we obtain the following system:

$$\frac{d\kappa_a}{ds_a} = 0, \quad \kappa_a \frac{d\kappa_a}{ds_a} = 0, \quad \frac{d^2\kappa_a}{ds_a^2} = 0,$$

which leads us to $\kappa_a =$ constant. Then S_q is a q-slant ruled surface in E^3. □

Example 4.12. Let the ruled surface S_q be given by the parameterization

$$\vec{r}(u,v) = \left(\frac{1}{3}(1+u)^{3/2} + v\frac{1}{2}(1+u)^{1/2}, \frac{1}{3}(1-u)^{3/2} - v\frac{1}{2}(1-u)^{1/2}, \frac{1}{\sqrt{2}}u + v\frac{1}{\sqrt{2}}\right),$$

(Fig. 1). After a simple calculation, we obtain

$$\vec{q}(s_q) = \left(\frac{1}{2}\left(1+\sin\left(\sqrt{8}s_q\right)\right)^{1/2}, -\frac{1}{2}\left(1-\sin\left(\sqrt{8}s_q\right)\right)^{1/2}, \frac{1}{\sqrt{2}}\right),$$

$$\vec{h}(s_q) = \left(\frac{\sqrt{2}\cos\left(\sqrt{8}s_q\right)}{2\sqrt{1+\sin\left(\sqrt{8}s_q\right)}}, \frac{\sqrt{2}\cos\left(\sqrt{8}s_q\right)}{2\sqrt{1-\sin\left(\sqrt{8}s_q\right)}}, 0\right),$$

$$\vec{a}(s_q) = \left(-\frac{\cos\left(\sqrt{8}s_q\right)}{2\sqrt{1-\sin\left(\sqrt{8}s_q\right)}}, \frac{\cos\left(\sqrt{8}s_q\right)}{2\sqrt{1+\sin\left(\sqrt{8}s_q\right)}}, \frac{1}{\sqrt{2}}\right),$$

and the conical curvature $\kappa(s_q) = 1 =$ constant. Therefore S_q is a q-slant ruled surface and it is easily seen that the conditions of theorems given above are satisfied.

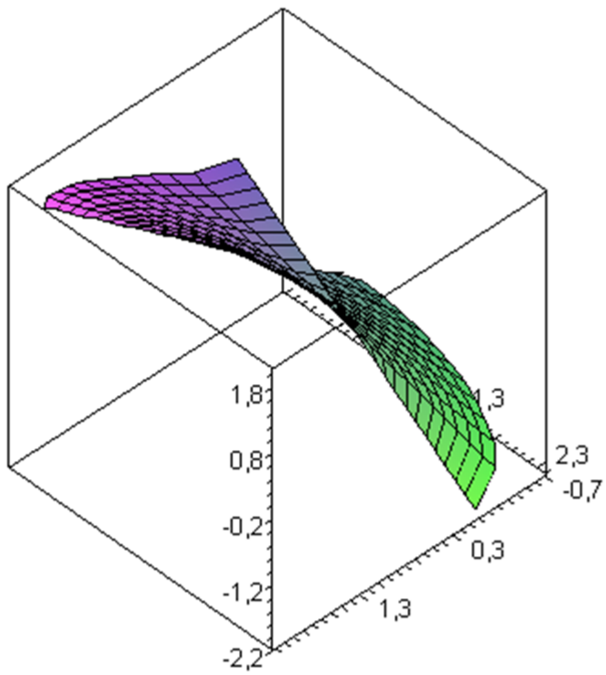

FIGURE 1. q-slant ruled surface S_q.

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