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Reduction of Differential Equations by Lie Algebra of Symmetries

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Abstract. The paper is devoted to an application of Lie group theory to differential equations. The basic infinitesimal method for calculating symmetry group is presented, and used to determine general symmetry group of some differential equations. We include a number of important applications including integration of ordinary differential equations and finding some solutions of partial differential equations together with some examples. A Bianchi theorem for the solvable symmetry groups is given to reduce a system of ordinary differential equations.

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Introduction

The symmetry groups that arises most often in the applications to geometry and differential equations are Lie groups of transformations acting on a finite-dimensional locally Euclidean space containing independent and dependent variables which has a manifold structure. Since Lie groups will be one of the essential tool in geometric theory of differential equatiopns, it is important that we gain a basic familiarity with these fundamental mathematoical objects. The first section of the paper is

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devoted to survey of an algorithmic method for finding an special kind of Lie groups called symmetry groups of a given system of differential equations which transform solutions to solutions. Such general transformation groups figure prominently in Lie's theory of symmetry groups of differential equations.

Since we are dealing with differential equations we must be able to handle the derivatives of the dependent variables on the some footing as the independent and dependent variables themselves. The proper geometric contex for these purposes the so-called "jet spaces", well known to nineteenth century practitioners, but first formally defined by Ehresmann [2]. But we skip out to discuss this concept in the paper and we can only say that it is a locally Euclidean space for Taylor expansion of all functions with associated independent and dependent variables. Lie group actions on jet spaces leads us to the concept of "prolonging" a group action in the space of independent and dependenta variables of the system. The key prolongation formula for an infinitesimal generator of a group of transformations, given in theorem 1.4, then provides the basis for the systematic determination of symmetry groups of differential equations. Application to some physical partial differential equations are presented in the sequel.

In the case of ordinary differential equations [7], Lie showed how knowledge of a one-parameter symmetry group allows us to reduce the order of equation by one. In particular, a first order equation with a known one-dimensional symmetry group can be integrated by a single quadrature. But, in the case of higher dimensional symmetry groups; it is not in general possible to reduce the order of an equation invariant under an $r-$dimensional symmetry group by r using only quadratures. A Bianchi theorem discuss how the theory proceeds for multi-dimensional symmetry groups for higher order equation and system of ordinary differential equations.

1. Mathematical Formulations

Consider a general $n-$th order system of differential equations

$$\Delta_\nu(\mathbf{x}, \mathbf{u}^{(n)}) = 0, \qquad \nu = 1, ..., m, \tag{1.1}$$

with $p-$independent variables $\mathbf{x} = (x^1, ..., x^p)$, and $q-$dependent variables $\mathbf{u} = (u^1, ..., u^q)$, with $\mathbf{u}^{(n)}$ denoting the derivatives of the $\mathbf{u}'s$ with respect to independent variables up to order n. The system of differential equations (1.1) which we often abbreviate as $\Delta = 0$, is thus defined by the vanishing of a collection of differential functions $\Delta_\nu : \mathbf{J}^n \to \mathbb{R}$ defined on the $n-$th jet space $\mathbf{J}^n$.

Definition 1.1. A *symmetry* of system of differential equations (1.1) means a transformation from its independent and dependent variables called *total space* E of (1.1) to itself which maps solutions to solutions.

The most basic type of symmetry is a group G of point transformations on the associated total space. It means for any $g \in G$ a point transformation $g : E \to E$ acting on E is called a symmetry of (1.1), if whenever $\mathbf{u} = \mathbf{f}(\mathbf{x})$ is a solution to (1.1), and the transformed function $\tilde{\mathbf{f}} = g \cdot \mathbf{f}$ is well-defined, then $\tilde{\mathbf{f}}$ is also a solution to (1.1).

Definition 1.2. An $n-$th *prolonagation* of the transformation $g : E \to E$ is a transformation $g^{(n)} : \mathbf{J}^n \to \mathbf{J}^n$ on the $n-$th jet space of E which acts on the derivatives of g up to order n additionally.

Suppose G is a Lie group acting on the total space E. This action is generated by a differential operator

$$X = \sum_{i=1}^{p} \xi^i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \varphi_\alpha(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u_\alpha}, \tag{1.2}$$

called the *infinitesimal generator* of the action. Each infinitesimal generator's flow coincides with the action of the corresponding one-parameter subgroup of G. Specifically, if $\tilde{X}$ is a generator of the corresponding Lie algebra $\mathcal{G}$ of the Lie group G generates the one-parameter subgroup $\{\exp(\epsilon\tilde{X}) | \epsilon \in \mathbb{R}\} \leq G$, then we identify $\tilde{X}$ with the infinitesimal generator X of the one-parameter group of transformations or flow $x \mapsto \exp(\epsilon\tilde{X})(x)$. According to differential geometry the infinitesimal generators of the group action are found by differentiation

$$X|_x = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \exp(\epsilon\tilde{X})(x), \qquad x \in E, \qquad \tilde{X} \in \mathcal{G}. \tag{1.3}$$

Thus, if $\theta^{(x)} : G \to E$ is the corresponding orbit map of the action G on E, then, the directional differentiation

$$X|_x = \mathrm{d}\theta^{(x)}(\tilde{X}|_e) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \theta^{(x)}(e + \epsilon\tilde{X}),$$

gives the infinitesimal generators. The following theorem shows that the set of infinitesimal generators of a given action is a Lie algebra.

Theorem 1.3. *Let $\mathcal{G}$ be a finite-dimensional Lie algebra of differential operators (1.2) on E. Let G denote a Lie group having Lie algebra $\mathcal{G}$. Then there is a local action of G whose infinitesimal generators coincide with the given Lie algebra.*

Given a differential operator X generating a one-parameter group of transformations $\exp(\epsilon X)$ on E, the associated $n-$th order prolonged differential operator $X^{(n)}$ is a differential operator on the jet space $\mathbf{J}^n$ which is the infinitesimal generator of the prolonged one-parameter group $\exp(\epsilon X)^{(n)}$. Thus at any point $(\mathbf{x}, \mathbf{u}^{(n)}) \in \mathbf{J}^n$,

$$X^{(n)}|_{(\mathbf{x},\mathbf{u}^{(n)})} = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \exp(\epsilon X)^{(n)} \cdot (\mathbf{x}, \mathbf{u}^{(n)}). \qquad (1.4)$$

The explicit formula for the prolonged differential operator is provided by the following prolongation formula. Although, the formula can be proved by direct computation based on the definition (1.4), [8].

Theorem 1.4. *Suppose X be a differential operator given by (1.2), and let $\mathbf{Q} = (Q^1, ..., Q^q)$ be a $q-$tuple of differential functions given by*

$$Q^\alpha(\mathbf{x}, \mathbf{u}^{(1)}) = \varphi_\alpha(\mathbf{x}, \mathbf{u}) - \sum_{i=1}^{p} \xi_i(\mathbf{x}, \mathbf{u}) \frac{\partial u^\alpha}{\partial x_i}. \qquad (1.5)$$

The $n-th$ prolongation of X is given explicitly by

$$X^{(n)} = \sum_{i=1}^{p} \xi^i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{\sharp J=j=0}^{n} \varphi_\alpha^J(\mathbf{x}, \mathbf{u}^{(j)}) \frac{\partial}{\partial u_J^\alpha}, \qquad (1.6)$$

with coefficients

$$\varphi_\alpha^J = D_J Q^\alpha + \sum_{i=1}^{p} \xi_i u_{J,i}^\alpha, \qquad (1.7)$$

where D_J is the total derivative operator.

Definition 1.5. A system of differential equations is called *locally solvable* at each point $(\mathbf{x}_0, \mathbf{u}_0^{(n)})$, where $\Delta(\mathbf{x}_0, \mathbf{u}_0^{(n)}) = 0$, if there exist a smooth solution $\mathbf{u} = \mathbf{f}(\mathbf{x})$, defined in a neighborhood of $\mathbf{x}_0$, which achives the values of the indicated derivatives there: $\mathbf{u}_0^{(n)} = \mathbf{f}^{(n)}(\mathbf{x}_0)$. System (1.1) is a *regular* system if its Jacobian is of maximal rank m at each point $(\mathbf{x}_0, \mathbf{u}_0^{(n)})$. If the conditions solvability and regularity are both satisfy, then, the system is called *fully regular*.

According to the definition 1.1 a transformation g is a symmetry of a locally solvable system of differential equations (1.1) if and only if the set of solutions is invariant under the prolongen transformation $g^{(n)}$.

1.1. **Infinitesimal Method.** We will henceforth assume that we are dealing with a connected Lie group of point transformation G. In this case the infinitesimal generators form a Lie algebra $\mathcal{G}$ consisting of differential operators (1.2) on the total space E. A fundamental theorem is a criterion for when an infinitesimal generator in the form (1.2) being a symmetry for the system (1.1):

Theorem 1.6. *A connected Lie group of transformations G is a symmetry group of the fully regular system of differential equations (1.1) if and only if the classical infinitesimal symmetry conditions*

$$X^{(n)}(\Delta_\nu) = 0, \qquad \nu = 1, ..., m \qquad \textit{whenever} \qquad \Delta = 0, \tag{1.8}$$

hold for every infinitesimal generator $X \in \mathcal{G}$ of G.

The condintions (1.8) are known as the *determining equations* of the symmetry group for the system. We now illustrate the practical use of the infinitesimal symmetry criterion (1.8) for determining the full (connected) symmetry group of several concrete differential equations of interests in some examples [5, 6].

1.1.1. *Born-Infeld Equation.* In physics, the Born-Infeld theory is a nonlinear generalization of electromagnetism. The model is named after physicists Max Born (1882-1970) and Leopold Infeld (1898-1968) who first proposed it. In physics, it is a particular example of what is usually known as a nonlinear electrodynamics. It was historically introduced in the 30's to remove the divergence of the electron's self-energy in classical electrodynamics by introducing an upper bound of the electric field at the origin. The Born-Infeld electrodynamics possesses a whole series of physically interesting properties: First of all the total energy of the electromagnetic field is finite and the electric field is regular everywhere. Second it displays good physical properties concerning wave propagation, such as the absence of shock waves and birefringence. A field theory showing this property is usually called completely exceptional and Born-Infeld theory is the only completely exceptional regular nonlinear electrodynamics. Finally (and more technically) Born-Infeld theory can be seen as a covariant generalization of Mie's theory, and very close to Einstein's idea of introducing a non-symmetric metric tensor with the symmetric part corresponding to the usual metric tensor and the antisymmetric to the electromagnetic field tensor. During the 1990 there was a revival of interest on Born-Infeld theory and its nonabelian extensions as they were found in some limits of string theory.

The equation is a second order non-linear partial differential equation of the form

$$\Delta_{BI} := (1-u_t^2)u_{xx} + 2u_x u_t u_{xt} - (1+u_x^2)u_{tt} = 0, \tag{1.9}$$

where u is a smooth function of (x,t). An infinitesimal point symmetry of equation (1.9) will be a differential operator $X = \xi^1(x,t,u)\partial_x + \xi^2(x,t,u)\partial_t + \varphi(x,t,u)\partial_u$, where $\partial_x = \partial/\partial_x$, etc. The infinitesimal symmetry criterion (1.8) is

$$\begin{aligned} &(1-u_t^2)\varphi^{xx} + 2(u_t u_{xt} - u_x u_{tt})\varphi^x + 2(u_x u_{xt} - u_t u_{xx})\varphi^t \\ &+2u_x u_t \varphi^{xt} - (1+u_x^2)\varphi^{tt} = 0, \qquad \text{whenever} \qquad \Delta_{BI} = 0. \end{aligned} \tag{1.10}$$

The coefficients $\varphi^x, \varphi^t, ...$ in (1.10) are obtained by (1.7). Thus

$$\begin{array}{ll} \varphi^x = D_x Q + \xi^1 u_{xx} + \xi^2 u_{xt}, & \varphi^t = D_t Q + \xi^1 u_{xt} + \xi^2 u_{tt}, \\ \varphi^{xx} = D_x^2 Q + \xi^1 u_{xxx} + \xi^2 u_{xxt}, & \varphi^{tt} = D_t^2 Q + \xi^1 u_{xtt} + \xi^2 u_{ttt}, \\ \varphi^{xt} = D_x D_t Q + \xi^1 u_{xxt} + \xi^2 u_{xtt}, & \end{array}$$

where

$$\begin{aligned} D_x &= \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + u_{xt}\partial_{u_t} + u_{xxx}\partial_{u_{xx}} + u_{xxt}\partial_{u_{xt}} + u_{xtt}\partial_{u_{tt}}, \\ D_t &= \partial_t + u_t\partial_u + u_{xt}\partial_{u_x} + u_{tt}\partial_{u_t} + u_{xxt}\partial_{u_{xx}} + u_{xtt}\partial_{u_{xt}} + u_{ttt}\partial_{u_{tt}}, \end{aligned}$$

are total derivative operators. Theorem 1.6 yields the over determining system of partial differential equations

$$\begin{array}{lllll} {\xi^2}_{xx} = 0, & {\xi^2}_{xu} = 0, & {\xi^2}_{tt} = 0, & {\xi^2}_{uu} = 0, & {\xi^1}_x = {\xi^2}_t, \\ {\xi^1}_t = {\xi^2}_x, & {\xi^1}_u = -\varphi_x, & {\xi^2}_t = \varphi_u, & {\xi^2}_u = \varphi_t, & {\xi^2}_{tu} = -\varphi_{xx}. \end{array}$$

The general solution to the determining system is readily found:

$$\begin{array}{ll} \xi^1 = c_1 + c_4 t - c_5 u + c_7 x, & \xi^2 = c_2 + c_4 x + c_6 u + c_7 t, \\ \varphi = c_3 + c_5 x + c_6 t + c_7 t, & \end{array}$$

where $c_1, ..., c_7$ are arbitrary constants. Thus the Lie algebra $\mathcal{G}$ of the Born-Infeld equation is spanned by the seven differential operators

$$\begin{array}{lll} X_1 = \partial_x, & X_2 = \partial_t, & X_3 = \partial_u, \\ X_4 = t\partial_x + x\partial_t, & X_5 = -u\partial_x + x\partial_u, & X_6 = u\partial_t + t\partial_u, \\ X_7 = x\partial_x + t\partial_t + u\partial_u. & & \end{array} \tag{1.11}$$

Table shows that these seven differential operators are closed under commutator bracket $[X_i, X_j] = X_i X_j - X_j X_i$, thus, they make a seven-dimansional real Lie algebra

Table 1. Commutation relations of $\mathcal{G}$

[,]	X_1	X_2	X_3	X_4	X_5	X_6	X_7
X_1	0	0	0	X_3	X_2	0	X_1
X_2	0	0	0	X_1	0	X_3	X_2
X_3	0	0	0	0	$-X_1$	X_2	X_3
X_4	$-X_2$	$-X_1$	0	0	$-X_6$	X_5	0
X_5	$-X_3$	0	X_1	X_6	0	X_4	0
X_6	0	$-X_3$	$-X_2$	$-X_5$	$-X_4$	0	0
X_7	$-X_1$	$-X_2$	$-X_3$	0	0	0	0

1.1.2. *Telegraph Equation.* The telegrapher's equations (or just telegraph equations) are a pair of linear differential equations which describe the voltage and current on an electrical transmission line with distance and time. The equations come from *Oliver Heaviside* who developed the transmission line model. The model demonstrates that the electromagnetic waves can be reflected on the wire, and that wave patterns can appear along the line. The cylindrical telegrapher's equations [10],

$$u_{tt} + ku_t = a^2\Big[\frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}\Big], \tag{1.12}$$

can be understood as a simplified case of Maxwell's equations. In a more practical approach, one assumes that the conductors are composed of an infinite series of two-port elementary components, each representing an infinitesimally short segment of the transmission line.

The Lie algebra of infinitesimal symmetries is the set of differential operators in the form of $X = \xi^1\partial_r + \xi^2\partial_\theta + \xi^3\partial_z + \xi^4\partial_t + \eta\partial_u$. This differential operator has the second prolongation

$$X^{(2)} = X + \varphi^r\partial_r + \varphi^\theta\partial_\theta + \varphi^z\partial_z + \varphi^t\partial_t + \cdots + \varphi^{zz}\partial_{u_{zz}} + \varphi^{zt}\partial_{u_{zt}} + \varphi^{tt}\partial_{tt}$$

with the coefficients

$$
\begin{aligned}
\varphi^{r} &= D_rQ+\xi^1u_{rr}+\xi^2u_{r\theta}+\xi^3u_{rz}+\xi^4u_{rt},\\
\varphi^{\theta} &= D_\theta Q+\xi^1u_{r\theta}+\xi^2u_{\theta\theta}+\xi^3u_{\theta z}+\xi^4u_{\theta t},\\
\varphi^{z} &= D_zQ+\xi^1u_{rz}+\xi^2u_{\theta z}+\xi^3u_{zz}+\xi^4u_{zt},\\
\varphi^{t} &= D_tQ+\xi^1u_{rt}+\xi^2u_{\theta t}+\xi^3u_{\theta t}+\xi^4u_{tt},\\
\varphi^{rr} &= D_r^2Q+\xi^1u_{rrr}+\xi^2u_{rr\theta}+\xi^3u_{rrz}+\xi^4u_{rrt},\\
\varphi^{rx} &= D_rD_\theta Q+\xi^1u_{r\theta r}+\xi^2u_{r\theta\theta}+\xi^3u_{r\theta z}+\xi^4u_{r\theta t},\\
\varphi^{rz} &= D_rD_zQ+\xi^1u_{rzr}+\xi^2u_{r\theta z}+\xi^3u_{rzz}+\xi^4u_{rzt},\\
\varphi^{rt} &= D_rD_tQ+\xi^1u_{rtr}+\xi^2u_{r\theta t}+\xi^3u_{rzt}+\xi^4u_{rtt},\\
\varphi^{\theta\theta} &= D_\theta^2Q+\xi^1u_{\theta\theta r}+\xi^2u_{\theta\theta\theta}+\xi^3u_{\theta\theta z}+\xi^4u_{\theta\theta t},\\
\varphi^{\theta z} &= D_\theta D_zQ+\xi^1u_{\theta zr}+\xi^2u_{\theta\theta z}+\xi^3u_{\theta zz}+\xi^4u_{\theta zt},\\
\varphi^{\theta t} &= D_\theta D_tQ+\xi^1u_{\theta rt}+\xi^2u_{\theta\theta t}+\xi^3u_{\theta zt}+\xi^4u_{\theta tt},\\
\varphi^{zz} &= D_z^2Q+\xi^1u_{rzz}+\xi^2u_{\theta zz}+\xi^3u_{zzz}+\xi^4u_{zzt},\\
\varphi^{zt} &= D_zD_tQ+\xi^1u_{rzt}+\xi^2u_{\theta zt}+\xi^3u_{zzt}+\xi^4u_{ztt},\\
\varphi^{tt} &= D_t^2Q+\xi^1u_{rtt}+\xi^2u_{\theta tt}+3u_{ztt}+\xi^4u_{ttt},
\end{aligned}
$$

where the operators D_r, D_θ, D_z and D_t denote the total derivative with respect to r, θ, z and t:

$$
\begin{aligned}
D_r &= \partial_r+u_r\partial_u+u_{rr}\partial_{u_r}+u_{r\theta}\partial_{u_\theta}+\cdots,\\
D_\theta &= \partial_\theta+u_\theta\partial_u+u_{\theta\theta}\partial_{u_\theta}+u_{r\theta}\partial_{u_r}+\cdots,\\
D_z &= \partial_z+u_z\partial_u+u_{zz}\partial_{u_z}+u_{rz}\partial_{u_r}+\cdots,\\
D_t &= \partial_t+u_t\partial_u+u_{tt}\partial_{u_t}+u_{rt}\partial_{u_r}+\cdots,
\end{aligned}
$$

Using the invariance condition (1.8), vanishes the second prolongation $X^{(2)}$ applied to equation (1.12), the following system of 27 determining equations are obtained:

$\xi^2{}_u=0,$
$\xi^3{}_u=0,$
$\xi^4{}_{rr}=0,$
$\xi^4{}_{rz}=0,$
$k\xi^4{}_z+2\eta_{ru}=0,$
$\xi^2{}_z+r\xi^2{}_{rz}=0,$
$2\xi^2{}_r+r\xi^2{}_{rr}=0,$
$\xi^3{}_t-a^2\xi^4{}_z=0,$
$r^2\xi^2{}_t-a^2\xi^4{}_\theta=0,$
$a^2r^2\eta_{rr}-kr^2\eta_t+a^2r\eta_r+a^2r^2\eta_{zz}$
$-r^2\eta_{tt}+a^2\eta_{\theta\theta}=0.$

$\xi^2{}_{zz}=0,$
$\xi^4{}_t=0,$
$\xi^4{}_{\theta z}=0,$
$\eta_{tu}=0,$
$\xi^1+r\xi^2{}_\theta=0,$
$\xi^2{}_{\theta\theta}-r\xi^2{}_r=0,$
$\xi^3{}_r-r\xi^2{}_{\theta z}=0,$
$\xi^4{}_\theta-r\xi^4{}_{r\theta}=0,$
$\xi^4{}_\theta+2\eta_{u\theta}=0,$

$\xi^3{}_z=0,$
$\xi^4{}_u=0,$
$\xi^4{}_{zz}=0,$
$\eta_{uu}=0$
$\xi^2{}_\theta+r\xi^2{}_{r\theta}=0,$
$k\xi^4{}_z+2\eta_{zu}=0,$
$\xi^3{}_\theta+r^2\xi^2{}_z=0,$
$\xi^4{}_{\theta\theta}+r\xi^4{}_r=0,$

The solution of the above system gives the following coefficients of the differential operator X:

$$\begin{aligned}
\xi^1 &= c_6\sin\theta - c_7\cos\theta - c_8 z\cos\theta - c_9 z\sin\theta + 2c_{10}a^2 t\sin\theta - 2c_{11}a^2 t\cos\theta,\\
\xi^2 &= c_1 + c_6 r^{-1}\cos\theta + c_7 r^{-1}\sin\theta + c_8 z r^{-1}\sin\theta - c_9 z r^{-1}\sin\theta\\
&\qquad +2c_{10}a^2 t r^{-1}\cos\theta - 2c_{11}a^2 t r^{-1}\sin\theta,\\
\xi^3 &= c_2 + 2c_5 a^2 t + c_8 r\cos\theta + c_9 r\sin\theta,\\
\xi^4 &= c_3 + 2c_5 a t + 2c_{10} r\sin\theta - 2c_{11} r\cos\theta,\\
\eta &= c_4 u - c_5 k z u - c_{10} k r u\sin\theta + c_{11} k r u\cos\theta,
\end{aligned}$$

where $c_1, \ldots, c_{11}$ are arbitrary constants, thus the Lie algebra $\mathcal{G}$ of the telegraph equation is spanned by the eleven differential operators

$$\begin{aligned}
&X_1 = \partial_\theta, \qquad X_2 = \partial_z, \qquad X_3 = \partial_t, \qquad X_4 = u\partial_u,\\
&X_5 = 2a^2 t\partial_z + 2z\partial_t - kzu\partial_u, \qquad X_6 = \sin\theta\partial_r + r^{-1}\cos\theta\partial_\theta,\\
&X_7 = -\cos\theta\partial_r + r^{-1}\sin\theta\partial_\theta,\\
&X_8 = -z\cos\theta\partial_r + r^{-1}z\sin\theta\partial_\theta + r\cos\theta\partial_z,\\
&X_9 = -z\sin\theta\partial_r - r^{-1}z\cos\theta\partial_\theta + r\sin\theta\partial_z,\\
&X_{10} = 2a^2 t\sin\theta\partial_r + 2a^2 t r^{-1}\cos\theta\partial_\theta + 2r\sin\theta\partial_t - kru\sin\theta\partial_u,\\
&X_{11} = -2a^2 t\cos\theta\partial_r + 2a^2 t r^{-1}\sin\theta\partial_\theta - 2r\cos\theta\partial_t + kru\cos\theta\partial_u,
\end{aligned}$$

An straightforward calculation shows that these eleven differential operators are closed under commutator bracket and thus form an eleven-dimansional real Lie algebra [4].

2. Invariant Functions

Given a group of point transformations G acting on total space E, the charachteristic of all $G-$invariant functions $\mathbf{u} = \mathbf{f}(\mathbf{x})$ is of great importance.

Definition 2.1. A function $\mathbf{u} = \mathbf{f}(\mathbf{x})$ is said to be *invariant* under the group transformation G if its graph $\{(\mathbf{x}, \mathbf{f}(\mathbf{x}))\}$ is a (locally) $G-$invariant subset.

For example, the graph of any invariant function for the rotation group SO(2) must be an arc of a circle centered at the origin, so $u = \pm\sqrt{c^2 - x^2}$.

The fundamental feature of Lie groups is the ability to work infinitesimally, thereby effectively linearizing complicated invariance criteria.

Theorem 2.2. *Let G be a connected Lie group of transformations acting on total space E. A function $I : E \to \mathbb{R}$ is invariant under G if and only if for all $(\mathbf{x}, \mathbf{u}) \in E$ and every infinitesimal generator $X \in \mathcal{G}$ of G,*

$$X[I(\mathbf{x}, \mathbf{u})] = 0. \tag{2.1}$$

Thus, according to theorem 2.2, the invariant $v = I(\mathbf{x}, \mathbf{u})$ of a one-dimensional group with infinitesimal generator (1.2), obtained from (2.1), satisfy the first order, linear, homogeneous partial differential equation

$$\sum_{i=1}^{p} \xi^i(\mathbf{x}, \mathbf{u}) \frac{\partial v}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi_\alpha(\mathbf{x}, \mathbf{u}) \frac{\partial v}{\partial u^\alpha} = 0. \tag{2.2}$$

The solutions of (2.2) are effectively found by the method of charachteristics. We replace the partial differential equation by the charachteristic system of ordinary differential equations

$$\frac{dx^1}{\xi^1(\mathbf{x}, \mathbf{u})} = \cdots = \frac{dx^p}{\xi^p(\mathbf{x}, \mathbf{u})} = \frac{du^1}{d\varphi_1(\mathbf{x}, \mathbf{u})} = \cdots = \frac{du^q}{\varphi_q(\mathbf{x}, \mathbf{u})}. \tag{2.3}$$

The general soluition to (2.3) can be written in the form $I_1(\mathbf{x}, \mathbf{u}) = c_1, ..., I_{p+q-1}(\mathbf{x}, \mathbf{u})) = c_{p+q-1}$, where c_i are constants of integrations.

Lemma 2.3. *The resulting functions $I_1, ..., I_{p+q-1}$ form a complete set of functionally independent invariants of the one-dimensional Lie algebra spanned by differential operator X.*

For example a one-dimensional Lie algebra spanned by the differential operator $X = -y\partial_x + x\partial_y + (1 + z^2)\partial_z$ are obtained by solving the charachteristic system

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{dz}{1 + z^2},$$

thus, there are two functionally independent invariant functions $r = \sqrt{x^2 + y^2}$ and $w = (xz - y)/(yz + x)$. A fundamental theorem obtained from differential geometry charachterizes the number of functionally independent invariants of a group action.

Theorem 2.4. *Let G be a transformation group acting semi-regularly (all the orbits have same dimension) on total space E with s-dimensional orbits. Let $I_1(\mathbf{x}, \mathbf{u}), ..., I_{p-s}(\mathbf{x}, \mathbf{u}), J_1(\mathbf{x}, \mathbf{u}), ..., J_q(\mathbf{x}, \mathbf{u})$, be a complete set of functionally independent invariants for G. Then any G-invariant function $\mathbf{u} = \mathbf{f}(\mathbf{x})$, can locally be written in the implicit form*

$$\mathbf{w} = \mathbf{h}(\mathbf{y}), \qquad \textit{where} \qquad \mathbf{y} = \mathbf{I}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{w} = \mathbf{J}(\mathbf{x}, \mathbf{u}). \tag{2.4}$$

Remark 2.5. A "similarity solution" or "invariant solution" which is a main subject of next section, of a system of partial differential equations is just an invariant function for a group of scalling transformations. For example, consider the one-dimensional group $\mathbb{R}^+$ acting on $\mathbb{R}^3$ with the transformation $(x, y, u) \mapsto (\lambda x, \lambda^\alpha y, \lambda^\beta u)$. The independents invariants are provided by the rotios $y = y/x^\alpha, w = u/x^\beta$, so any scale-invariant function can be written as $w = h(y)$, or explicitely $u = x^\beta h(y/x^\alpha)$.

As usual, the most convenient charachterization of the invariant functions is based on an infinitesimal conditions. Since the graph of a function is defined by the vanishing of its components $u^\alpha - f^\alpha(\mathbf{x})$, the general invariance theorem 1.6 imposes the infinitesimal invariance conditions

$$0 = X(u^\alpha - f^\alpha(\mathbf{x})) = \varphi^\alpha(\mathbf{x}, \mathbf{u}) - \sum_{i=1}^{p} \xi^i(\mathbf{x}, \mathbf{u}) \frac{\partial f^\alpha}{\partial x^i},$$

which must hold whenever $\mathbf{u} = \mathbf{f}(\mathbf{x})$, for every infinitesimal generator $X \in \mathcal{G}$, as in (1.2). These first order partial differential equations are known in the literature as the invariant surface conditions associated with the given transformation group.

3. Group-Invariant Solutions

When we confronted with a complicated system of partial differential equations in some physically important problem, the discovery of any explicit solutions whatsoever is of great interest. Explicit solutions can be used as models for physical experiments, as benchmarks for testing numerical methods, etc., and often reflect the asymptotic or dominant bahaviour of more general types of solutions. The method used to find group-invariant solutions, generalizing the well-known techniques for finding similarity solutions, provide a systematic computational method for determining large classes of special solutions. These group-invariant solutions are characterized by their invariance under some symmetry group of the system of partial differential equations; the more symmetrical the solution, the easier it is to construct. The fandamental theorem on group-invariant solutions roughly states that the solutions which are invariant under a given $r-$dimensional symmetry group of the system can all the find by solving r fewer independent variables than the original system. In particular, if the number of parameters is one less that the number of independent variables in the physical system: $r = p - 1$, then all the corresponding group-invariamt solutions can be found by solving a system of **ordinary differential equations.** In this way, one reduces an intractable set of partial differential equations to a simpler set of ordinary differential equations which one might stand a chance of solving explicitly. In practical applications, these group-invariant solutions can, in most instances, be effectively found and, often, are the only explicit solutions which are known.

3.1. **Construction of Group-Invariant Solutions.** Consider a system of partial differential equations (1.1) with $p-$independent and $q-$dependent variables. Let G be a group of transformations acting on

E. A solution $\mathbf{u} = \mathbf{f}(\mathbf{x})$ of the system is said to be *G-invariant* if it is left unchanged by all the group transformations in G, meaning that for each $g \in G$, the function $\mathbf{f}$ and $g \cdot \mathbf{f}$ agree on their common domains of definition.

If G is a symmetry group of a system of partial differential equations (1.1), then, we can find all the $G-$invariant solutions to Δ by solving a reduced system of differential equations, denoted by Δ/G, which will involve fewer independent variables than the original system Δ. To see how this reduction effected, we begin by making the simplifying assumption that G acts *projectably* on M. This means that the transformations in G all takes the form $(\tilde{\mathbf{x}}, \tilde{\mathbf{u}}) = g \cdot (\mathbf{x}, \mathbf{u}) = (\Xi^g(\mathbf{x}), \Phi_g(\mathbf{x}, \mathbf{u}))$ for $g \in G$, i.e., the changes in the independent variable $\mathbf{x}$ do not depend on the dependent variables $\mathbf{u}$. There is then a projected group action $\tilde{\mathbf{x}} = g \cdot \mathbf{x} = \Xi_g(\mathbf{x})$ on an open sebset $\Omega \subset X$. We make the regularity assumption that both the action of G on E and the projected action of G on Ω is *regular*, i.e., all the orbit dimension of the action are same as s, where s is strictly less than p. (The case $s = p$ is fairly trivial, while if $s > p$, no $G-$invariant functions exist, Usually s will be the same as the dimension of G itself, but this need not be the case.) Under these assumption there exist $p - s$ functionally independent invariants $y^1 = \eta^1(\mathbf{x}), ..., y^{p-s} = \eta^{p-s}(\mathbf{x})$ of the projected group action on $\Omega \subset X$. Each of this functions is also an invariant of the full group action on E , and furthermore, we can find q additional invariants of the action of G on E, of the form $\mathbf{v}^1 = \zeta^1(\mathbf{x}, \mathbf{u}), ..., \mathbf{v}^q = \zeta^q(\mathbf{x}, \mathbf{u})$, which, together with the η's provide a complete set of $p+q-s$ functionally independent invariants for G on E. We write this complete collection on invariants as

$$\mathbf{y} = \eta(\mathbf{x}), \qquad \mathbf{v} = \zeta(\mathbf{x}, \mathbf{u}). \tag{3.1}$$

In the construction of the reduced system of differential equations for the $G-$invariant solutions to Δ, then $\mathbf{y}'$s will play the role of the new independent variables, and the $\mathbf{v}'$s the role of the new dependent variables. Note in particular that there are s few independent variables $y^1, ..., y^{p-s}$ which will appear in this reduced system, where s is the dimension of the orbits of G.

There is a one-to-one correspondence between $G-$invariant function $\mathbf{u} = \mathbf{f}(\mathbf{x})$ on E and arbitrary functions $\mathbf{v} = \mathbf{h}(\mathbf{y})$ involving the new variables. To explain this correspondence, we begin by invoking the implicit function theorem to solve the system $\mathbf{y} = \eta(\mathbf{x})$ for $p - s$ of the independent variables, say $\tilde{\mathbf{x}} = (x^{i_1}, ..., x^{i_{p-s}})$, in the terms of the new variables $y^1, ..., y^{p-s}$ and the remaining s old independent variables, denoted as

$\hat{\mathbf{x}} = (x^{j_1}, ..., x^{j_s})$. Thus we have the solutions

$$\tilde{\mathbf{x}} = \rho(\hat{\mathbf{x}}, \mathbf{y}), \tag{3.2}$$

for some well-defined function ρ. The first $p - s$ of the old independent variables $\tilde{\mathbf{x}}$ are known as *principle variables*, and the remaining s of these variables $\hat{\mathbf{x}}$ are the *parametric variables*, as they will, enter parametrically into all the subsequent fromulae. The precise manner in which one splits the variables $\mathbf{x}$ into principle and parametric variables is restricted only by the requirement that the $(p-s) \times (p-s)$ submatris $(\partial \eta^j / \partial \tilde{x}^i)$ of the full Jacobian matrix $\partial \eta / \partial \mathbf{x}$ is invertible, so that the implicit function theorem is applicable; otherwise, the choice is entirely arbitrary. We need to make a further transversality assumption on the action of G on E, that allows us to solve the other system of invariants $\mathbf{v} = \zeta(\mathbf{x}, \mathbf{u})$ for all the dependent variables $u^1, ..., u^q$ in terms of $x^1, ..., x^p$, and $v^1, ..., v^q$, and hence in terms of new variables $\mathbf{y}, \mathbf{v}$ and parametric variables $\hat{\mathbf{x}}$:

$$\mathbf{u} = \tilde{\mu}(\mathbf{x}, \mathbf{v}) = \tilde{\mu}(\hat{\mathbf{x}}, \rho(\hat{\mathbf{x}}, \mathbf{y}), \mathbf{v}) = \mu(\hat{\mathbf{x}}, \mathbf{y}, \mathbf{v}), \tag{3.3}$$

near any point $(\mathbf{x}_0, \mathbf{u}_0) \in E$.

If $\mathbf{v} = \mathbf{h}(\mathbf{y})$ is any smooth function, then (3.3) coupled with (3.1) produces a corresponding $G-$invariant function on E, of the form

$$\mathbf{u} = \mathbf{f}(\mathbf{x}) = \mu(\hat{\mathbf{x}}, \eta(\mathbf{x}), \mathbf{h}(\eta(\mathbf{x}))). \tag{3.4}$$

Conversely, if $\mathbf{u} = \mathbf{f}(\mathbf{x})$ is any $G-$invariant function on E, then it is not too difficult to see that there necessarily exist a function $\mathbf{v} = \mathbf{h}(\mathbf{y})$ such that $\mathbf{f}$ and the corresponding function (3.4) locally agree. Thus, we have seen how $G-$invariance of functions serves to decrease the number of variables upon which they depend.

We are now interested in finding all the $G-$invariant solutions to some system of partial differential equations (1.1). In other words, we want to know when a function of the form (3.4) corresponding to a function $\mathbf{v} = \mathbf{h}(\mathbf{y})$ is a solution to Δ. This will impose certain constrains on the function $\mathbf{h}$; these are found by computing the formulae for the derivatives of $\mathbf{v} = \mathbf{h}(\mathbf{y})$ with respect to $\mathbf{y}$, and then substituting these into the system of differential equations Δ.Thus we need to know how the derivatives of the functions $\mathbf{v} = \mathbf{h}(\mathbf{y})$ are related to the derivatives of the corresponding $G-$invariant function $\mathbf{u} = \mathbf{f}(\mathbf{x})$. However, this is an easy application of the chain rule. Differentiating (3.4) with respect to $\mathbf{x}$ leads to a system equation of the form

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}} \left[\mu(\hat{\mathbf{x}}, \mathbf{y},) \right] = \frac{\partial \mu}{\partial \hat{\mathbf{x}}} + \frac{\partial \mu}{\partial \mathbf{y}} \frac{\partial \eta}{\partial \mathbf{x}} + \frac{\partial \mu}{\partial \mathbf{v}} \frac{\partial \mathbf{v}}{\partial \mathbf{y}} \frac{\partial \eta}{\partial \mathbf{x}},$$

since $\mathbf{y} = \eta(\mathbf{x})$. Here, $\partial \mathbf{u} / \partial \mathbf{x}$, etc., denoted Jacobian matrices of first order derivatives of indicated variables. Moreover, using (3.2), we can rewrite $\partial\eta/\partial\mathbf{x}$ in terms of $\mathbf{y}$ and parametric variables $\hat{\mathbf{x}}$, Thus we obtain an equation of the form

$$\frac{\partial\mathbf{u}}{\partial\mathbf{x}} = \mu_1\left(\hat{\mathbf{x}}, \mathbf{y}, \mathbf{v}, \frac{\partial\mathbf{v}}{\partial\mathbf{y}}\right),$$

expressing the first order derivatives of any $G-$invariant function $\mathbf{u}$ with respect to $\mathbf{x}$ in terms of $\mathbf{y}, \mathbf{v}$, the first order derivatives of $\mathbf{v}$ with respect to $\mathbf{y}$ together with parametric variables $\hat{\mathbf{x}}$. Continuing to differentiate using the chain rule, and substituting to (3.2) whenever necessary, we are led to general formulae

$$\mathbf{u}^{(n)} = \mu^{(n)}(\hat{\mathbf{x}}, \mathbf{y}, \mathbf{v}^{(n)}),$$

for all the derivatives of such a $\mathbf{u}$ up to order n with respect to $\mathbf{x}$ in terms of $\mathbf{y}, \mathbf{v}$, the derivatives of $\mathbf{v}$ with respect to $\mathbf{y}$ up to order n, and the ubiquitous parametric variable $\hat{\mathbf{x}}$.

Once the relevant formulae relating derivatives of $\mathbf{u}$ with respect to $\mathbf{x}$ to those of $\mathbf{v}$ with respect to $\mathbf{y}$ have been determined, the reduced system of differential equations for the $G-$invariant solutions to the system Δ is determined by substituting these expressions into the system whenever they occur. In general, this leads to system of differential equations of the form

$$\tilde{\Delta}_\nu\left(\hat{\mathbf{x}}, \mathbf{y}, \mathbf{v}^{(n)}\right), \qquad , \nu = 1, .., \ell,$$

still involving parametric variables $\hat{\mathbf{x}}$. If G is a symmetry group for Δ, this resulting system will in fact always be equivalent to a system of equations denoted by

$$(\Delta/G)_\nu(\mathbf{y}, \mathbf{v}^{(n)}) = 0, \qquad \nu = 1, ..., \ell,$$

which are independent of the parametric variables, and thus constitute a genuine system of differential equations for $\mathbf{v}$ as a function of $\mathbf{y}$. This is the reduced system Δ/G for the $G-$invariant solutions to the system Δ. Every solution $\mathbf{v} = \mathbf{h}(\mathbf{y})$ of Δ/G will corresponds, via (3.4), to a $G-$invariant solution to Δ, and moreover every $G-$invariant solution can be constructed in this manner.

3.2. **Examples of Group-Invarinat Solutions.** Before attempting to prove that the basic procedure for constructing group-invariant solutions outlined above works, we will illustrate the method with some systematic examples, constructing group invariant solutions for some physical partial differential equations.

The Heat Equation. The symmetry Lie algebra of the heat equation $u_t = u_{xx}$ are six differential operators

$$\begin{aligned} X_1 &= \partial_x, & X_2 &= \partial_t, \\ X_3 &= u\partial_u, & X_4 &= x\partial_x + 2t\partial_t, \\ X_5 &= 2t\partial_x - xu\partial_u, & X_6 &= 4tx\partial_x + 4t^2\partial_t - (x^2+2t)u\partial_u, \end{aligned}$$

plus an infinite-dimensional subalgebra

$$X_\alpha = \alpha(x,t)\partial_u,$$

where α is an arbitrary solution of the heat equation. For each one-parameter subgroup of the full symmetry group there will be a corresponding class of group-invariant solutions which will be determined from a reduced ordinary differential equation, whose form will in general depend on the particular subgroup under investigation.

The global invariant of the linear symmetry $\partial_t + c\partial_x$, in which c is a fixed constant, are

$$y = x - ct, \qquad v = u, \tag{3.5}$$

which correspond to translation group $(x,t,u) \mapsto (x + c\varepsilon, t + \varepsilon, u)$ for $\varepsilon \in \mathbb{R}$, so that the group-invariant solution $v = h(y)$ takes the familiar form $u = h(x - ct)$, determine a wave of unchanging profile moving at the constant velocity c. Solving for the derivatives of u with respect to x and t in terms of those of v with respect to y we find

$$u_t = -cv_y, \qquad u_x = v_y, \qquad u_{xx} = v_{yy},$$

and so on. Substituting these expressions into the heat equation, we find the reduced ordinary differential equation for the travelling wave solutions to be

$$-cv_y = v_{yy}.$$

The general solution of this linear equation, is $v(y) = k_1e^{-cy} + k_2$, for k_1, k_2 arbitrary constants. Substituting back according to (3.5), we find a group-invariant solution called *travelling wave solution* to the heat equation: $u(x,t) = k_1e^{-c(x-ct)} + k_2$.

Similarly for any $a \in \mathbb{R}$, consider a linear symmetry $x\partial_x + 2t\partial_t + 2au\partial_u$, of the infinitesimal generators, which corresponds to the scaling group $(x,t,u) \mapsto (\lambda x, \lambda^2 t, \lambda^{2a}u)$, for $\lambda \in \mathbb{R}^+$. The global invariant of this subgroup is provided by $y = x/\sqrt{t}$ and $v - t^{-a}u$. Solving for the derivatives of u in terms of v, we find $u_t = t^{a-1}(\frac{1}{2}yv_y + av)$ and $u_{xx} = t^{a-1}v_{yy}$. Substituting these expressions into the heat equation, we find

$$v_{yy} + \frac{1}{2}yv_y - av = 0,$$

which form the reduced equation for the scale-invarinat solutions. The solution of this linear ordinary differential equation can be written in terms of parabolic cylinder functions. Indeed, if we set $w = v\exp(\frac{1}{8}y^2)$, then w satisfies a scaled form of Weber's differential equation,

$$w_{yy} = \left[\left(a + \frac{1}{4}\right) + \frac{1}{16}y^2\right] w.$$

The general solution of the heat equation called scale-invariant solution is

$$u(x,t) = t^a e^{-\frac{x^2}{8t}} \left[k_1 \mathrm{U}\left(2a + \frac{1}{2}, \frac{x}{\sqrt{2t}}\right) + k_2 \mathrm{V}\left(2a + \frac{1}{2}, \frac{x}{\sqrt{2t}}\right)\right],$$

where $\mathrm{U}(b,z)$ and $\mathrm{V}(b,z)$ are parabolic cylinder functions.

Vector Maxwell Equations in Non-Linear Optics. The equation of the model consist of Maxwell's equations coupled to a single Lorentz oscillator governing the polarization field $\mathbf{P}$, in which the oscillator is driven by the electric field $\mathbf{E}$. The equation of the model in dimensionless physical variables have the form [11]:

$$B_t + E_z = 0, \tag{3.6}$$

$$D_t + B_z = 0, \tag{3.7}$$

$$D = E + \frac{E^{2\sigma+1}}{2\sigma+1} + P, \tag{3.8}$$

$$P_{tt} + P - \alpha E = 0, \tag{3.9}$$

P is polarization along $x-$axis and D is displacement cuurent in (3.8)-(3.9).

Introducing potentials ϕ and $\mathcal{A}$ for the electric and magnetic fields E and B:

$$E = \phi_z, \qquad B = \mathcal{A}_z. \tag{3.10}$$

Faraday's law (3.7) can be written as $(E + \mathcal{A}_t)_z = 0$. Thus

$$B = \mathcal{A}_z, \qquad E = -\mathcal{A}_t, \tag{3.11}$$

are representations for B and E in terms of magnetic potential $\mathcal{A}$. In this representation of the field E and B, Faraday's law (3.7) is automatically satisfied, as a consequence of the integrability condition $\mathcal{A}_{zt} = \mathcal{A}_{tz}$. Thus, (3.7)-(3.9) reduce to the system

$$\frac{\partial}{\partial t}\left(\mathcal{A}_t - \frac{\mathcal{A}_t^{2\sigma+1}}{2\sigma+1} + P\right) + \mathcal{A}_{zz} = 0, \tag{3.12}$$

$$P_{tt} + P + \alpha\mathcal{A}_t = 0. \tag{3.13}$$

Theorem (2.4) together with a straightforward calculations shows that the symmetries Lie algebra of the system (3.13)-(3.13) is spanned by the differential operaors

$$X_1 = \partial_t, \qquad X_2 = \partial_z, \qquad X_3 = z\partial_{\mathcal{A}}, \qquad X_4 = \partial_{\mathcal{A}}. \tag{3.14}$$

The invariants of the linear symmetry $X = \partial_t + \partial_z + (z+1)\partial_{\mathcal{A}}$ are $z - ct = y, \mathcal{A} - (\frac{1}{2}z^2 + z) = v_1$ and $P = v_2$. From (3.11), the solutions for E and B have the form

$$E = cf'(y), \qquad B = z + 1 + f'(y). \tag{3.15}$$

where $f(y)$ is a functional constant in the ansatz solution $\mathcal{A} = \frac{1}{2}z^2 + z + f(y)$. The solutions of for $P(y)$ and $f(y)$ depend on the traveling wave variable $y = z - ct$, where c is the velocity of the traveling wave fram. From (3.15), $E - cB = -c(1+z)$.

Substituting the ansatz solution into Ampere's law (3.13), and integrating with respect to y yields the integral:

$$y + \frac{E}{c} - c\left(E + \frac{E^{2\sigma+1}}{2\sigma+1} + P\right) = k, \tag{3.16}$$

where k is an integrating constant. The Lorentz oscillator equation (3.13) becomes:

$$c^2 P''(y) + P = \alpha E. \tag{3.17}$$

Hence the system reduces to a second order ordinary differential equation for P coupled with an algebraic equation (3.16).

3.2.1. *Born-Infeld Equation.* In (1.11) we find the seven-dimensional symmery group pf the Born-Infeld equation. In this section we will find all group-invariant solutions due to all seven symmetries.

a) *Space translation invariance X_1.* The invariants of this symmetry are $v(y) = u(x,t)$ and $y = t$, thus the reduced equation respect to this invariants is $v'' = 0$, and the relative group-invariant solution is $u = c_1 t + c_2$.
b) *Time translation invariance X_2.* The invariants of this symmetry are $v(y) = u(x,t)$ and $y = x$, thus the reduced equation respect to this invariants is $v'' = 0$, and the relative group-invariant solution is $u = c_1 x + c_2$.
c) *Solution translation invariance X_3.* For this symmetry every translated solution with any constant is a similarity solution.
d) *Hyperbolic rotation invariance on time and space X_4.* The invariants of this symmetry are $v(y) = u(x,t)$ and $y = -x^2 + t^2$,

thus the reduced equation respect to this invariants is

$$2(2y^2+e^{4\varepsilon}+2e^{-4\varepsilon}y^4)v'^2v''+4(e^{-2\varepsilon}y^2+e^{2\varepsilon})v'^3 \\ -4yv''-4v'=0,$$

and the relative group-invariant solution is

$$u=\pm c_1\arctan\left(\frac{x^2-t^2+2c_1^2}{\sqrt{(x^2-t^2)(-x^2+t^2-4c_1^2)}}\right)+c_2.$$

e) *Rotation invariance on solution and space* X_5. The invariants of this symmetry are $v(y)=x^2+u(x,t)^2$ and $y=-x^2+t^2$, thus the reduced equation respect to this invariants is

$$\cos(2\varepsilon)\cos^2\varepsilon v''-2\cos(2\varepsilon)v'^2+2\cos^2\varepsilon v=0,$$

and the relative group-invariant solutions are

$$u = \pm\frac{1}{2}\sqrt{2c_1e^{\frac{t+c_2}{c_1}}+8c_1^3e^{-\frac{t+c_2}{c_1}}-4x^2-8c_1^2},$$
$$u = \pm\frac{1}{2}\sqrt{2c_1e^{-\frac{t+c_2}{c_1}}+8c_1^3e^{\frac{t+c_2}{c_1}}-4x^2-8c_1^2}.$$

f) *Hyperbolic rotation invariance on time and solution* X_6. The invariants of this symmetry are $v(y)=-t^2+u(x,t)^2$ and $y=x$, thus the reduced equation respect to this invariants is

$$(e^{2\varepsilon}v+2v^2+e^{-2\varepsilon}v^3)v''+(2v-e^{-2\varepsilon}v^2-e^{2\varepsilon})v'^2-2e^{2\varepsilon}v \\ -2v^3-4v^2=0,$$

and the relative group-invariant solutions are

$$u = \pm\frac{1}{2}\sqrt{2c_1e^{\frac{x+c_2}{c_1}}+8c_1^3e^{-\frac{x+c_2}{c_1}}+4x^2+8c_1^2},$$
$$u = \pm\frac{1}{2}\sqrt{2c_1e^{-\frac{x+c_2}{c_1}}+8c_1^3e^{-\frac{x+c_2}{c_1}}+4x^2+8c_1^2}.$$

g) *Scale invariance on time, space and solution simultainously* X_7. The invariants of this symmetry are $v(y)=\frac{u(x,t)}{x}$ and $y=\frac{t}{x}$, thus the reduced equation respect to this invariants is $v''(2y^2v'^2-4yvv'+v^2+y^2-1)=0$, and the relative group-invariant solutions are $u=c_1x+c_2t$, and $u=\pm\sqrt{-x^2+t^2}$

3.2.2. *Boussinesq Equation.* The Bossinesq equation

$$u_{tt} + uu_{xx} + u_x^2 + u_{xxxx} = 0, \tag{3.18}$$

is a soliton partial differential equation, and arises as a model equation for the unidirectional propagation of solitary wave in shallow water, [8]. A straightforward calculation eventually yields the complete symmetry algebra of the equation (3.18) is spanned by three differential operaors $X_1 = \partial_x, X_2 = \partial_t$ and $X_3 = x\partial_x + 2t\partial_t - 2u\partial_u$.

a) *Space translation invariance* X_1*.* The invariants of this symmetry are $v(y) = u(x,t)$ and $y = t$, thus the reduced equation respect to this invariants is $v'' = 0$, and the relative group-invariant solution is $u = \varphi(f)$ where φ satisfies the second order ordinary differential equation $\varphi_{ff} = -\frac{1}{2}\varphi(f)^2 - c_1 f - c_2$ for a differentiable function f respect to x.

b) *Time translation invariance* X_2*.* The invariants of this symmetry are $v(y) = u(x,t)$ and $y = x$, thus the reduced equation respect to this invariants is $v'''' + vv'' + v^2 = 0$, and the relative group-invariant solution is $u = c_1 t + c_2$.

c) *Scalling on space, time and solution itself simultainously* X_3*.* The invariants of this symmetry are $v(y) = e^{-2\varepsilon}u(x,t)$ and $y = e^{-2\varepsilon}t$, thus the reduced equation respect to this invariants is

$$\begin{aligned} 16y^4v'''' + 208y^3v''' + (1 + 732y^2 + 4y^2v)v'' + 4y^2v'^2 \\ +22yvv' + 120v = 0, \end{aligned}$$

and the invariant solution is of the series form!!!

4. Integration of Ordinary Differential Equations

In the previous section we find out that any system of partial differential equations could be reduced to a system of ordinary differential equations by invariants obtained from its symmetry group. This sections shows that knowledge of a one-dimensional group of symmetries of an ordinary differential equation allows us to reduce its order by one. At the end a theorem of Bianchi, [1], states that if an ordinary differential equation admits a $r-$dimensional solvable symmetry group, then its solutions can be determined, by quadrature, from those to a reduced equation of order $n - r$.

Definition 4.1. Let X be a differential operaor on the total space of a system of differential equations. A function $\mathbf{u} = \mathbf{f}(\mathbf{x})$ is called nontangential provided X is nowhere tangent to the graph of $\mathbf{f}$.

Theorem 4.2. *Let $\Delta(x, \mathbf{u}^{(n)}) = 0$ be an $n-th$ order scalar ordinary differential equation admitting a regular one-dimensional symmetry group G. Then all nontangential solutions can be found by quadrature from the solutions to an ordinary differential equation $(\Delta/G)(x, \mathbf{u}^{(n-1)}) = 0$ of order $n-1$, called the symmetry reduced equation.*

Example 4.3. Consider a general homogeneous second order linear equation

$$(x^2+1)u_{xx} + \sin x u_x + x^3 u = 0.$$

This equation clearly admits the one-dimensional (but it is not full symmetry) scaling symmetry generated by $u\partial u$. According to the general reduction procedure, as long as $u \neq 0$, we can introduce the new variable $v = \ln u, y = x$, in terms of which the equation becomes

$$v_{xx} + v_x^2 + \frac{\sin x}{x^2+1} v_x + \frac{x^3}{x^2+1} = 0,$$

if we set $v_x = w$ the reduced equation becomes to the first order *Riccati* equation.

Theorem 4.4. *Suppose $\Delta = 0$ is an $n-th$ order ordinary differential equation admitting a symmetry group G. Let $H \leq G$ be one-dimensional subgroup. Then the $H-$reduced equation $\Delta/H = 0$ admits the quotient group G_H/H, where $G_H = \{g|gHg^{-1} \subset H\}$ is the normalizer subgroup, as a symmetry group.*

Example 4.5. Consider a second ordinary differerntial equation of the form

$$x^2 u_{xx} = (xu_x - u)^2, \tag{4.1}$$

with infinitesimal symmetries $X_1 = x\partial_x, X_2 = x\partial_u$. Since $[X_1, X_2] = X_2$, if we reduce with respect to X_2, then the resulting first order equation will retain a symmetry corresponding to X_1, and hence Theorem 4.4 guarantees that it can be integrated. If we set $v = u/x$ and $w = v_x = x^{-2}(xu_x - u)$, so that equation (4.1) reduces to $x^3 w_x = (x^2 w)^2 + 2x^2 w$. This equation admits a scaling symmetry generated by the reduced differential operaors $\tilde{X} = x\partial_x - 2w\partial_w$, which means that it is of homogeneous form and can be integrated. If we set $\tilde{y} = w, \tilde{v} = \ln x$, equation (4.1) reduces to a first order equation $w_y = -w[1 + (w^{-1} - y)^2]$.

4.1. **Bianchi Theorem.** An $r-$dimensional Lie group G is called *solvable* if there exist a sequence of subgroups

$$\{e\} = G_0 \subset G_1 \subset \cdots \subset G_{r-1} \subset G_r = G,$$

such that each G_i is normal subgroup of G_{i+1}.

Definition 4.6. Suupose $\mathcal{G}$ is a finite-dimensional Lie algebra. A subspace $\mathcal{H} \subset \mathcal{G}$ is called an ideal Lie subalgebra of $\mathcal{G}$ if it is a Lie algebra respect to the Lie bracket on $\mathcal{G}$ and $[\mathcal{H}, \mathcal{G}] \subset \mathcal{H}$.

Theorem 4.7. *Let G be a connected Lie group, and suppose $H \leq G$ is a connected Lie subgroup. Then H is normal subgroup of G if and only if $\mathcal{H}$ is an ideal in $\mathcal{G}$.*

Theorem 4.7 shows that there is a corresponding between normal Lie subgroups and ideal Lie subalgebras. This is equivalent to the requirement that there exist a sequence of subalgebras

$$\{0\} = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_{r-1} \subset \mathcal{G}_r = \mathcal{G},$$

such that each $\mathcal{G}_i$ is ideal subgalgebras of $\mathcal{G}_{i+1}$. A Bianchi theorem states that:

Theorem 4.8. *If an ordinary differential equation admits a $r-$dimensional solvable Lie symmetry group, then its solution can be determined, by quadratures from those to a reduced equation of order $n - r$.*

For example consider the third order equation

$$u_x^5 u_{xxx} = 3u_x^4 u_{xx}^2 + u_{xx}^3, \tag{4.2}$$

with three-dimensional solvable Lie algebra of symmetries spanned by the differential operator

$$X_1 = \partial_u, \qquad X_2 = \partial_x, \qquad X_3 = u\partial_x.$$

Thus, (4.2) can be solved by quadratures.

5. Conclusion

In this paper we introduce the foundations and some applications of Lie's theory of symmetry groups of differential equations. The basic infinitesimal method for calculating symmetry groups is presented and used to fine the general symmetry group of some particular differential equations. The method needs complicated calculation when the variables and order of the given system increases, thus, some mathematical packages such as **Maple** and **Mathematica** are useful for finding symmetry algebras. An important application of symmetries for reduction of the equations are given in the sequel.

References

[1] L. Bianchi, Lezioni Sulla Teoria dei Gruppi Continui Finiti di Transformazioni, Enrico Spoerri, Pisa, 1918.

[2] C. Ehresmann, *Introduction a la Theorie des Structures Infinitesimales et des Pseudo-Groupes de Lie*, Geometrie Differentielle, Colloq. Inter. dur Centre Nat. de la Recherche Scientifique, Strasbourg, 1953, 97-110.

[3] S. Reza Hejazi, *Lie Group Analysis, Hamitonian Equiations and Conservation Laws of Born-Infeld Equation*, Asian-european Journal of Mathematics, Vol. 7, No. 3 (2014), 1450040 (19 pages), World Scientific Publishing Company, DOI: 10.1142/S1793557114500405.

[4] Mehdi Nadjafikhah and S. Reza Hejazi, *Symmetry Analysis of Telegraph Equation*, Asian-european Journal of Mathematics, Vol. 4, No. 1 (2011) 117126, World Scientific Publishing Company, DOI: 10.1142/S1793557111000101.

[5] M. Nadjafikhah and S. R. Hejazi, *Symmetry analysis of cylindrical Laplace equation*, Balkan J. Geom. Appl. (2009), pp. 6374.

[6] L. V. Ovsiannikov, Group Analysis of Differential Equations, Academic Press, New york, 1982

[7] P. J. Olver, Application of Lie Groups to Differential Equations, Second Edition, Graduate Texts in Mathematics, Vol. 107, Springer-Verlage, New York 1993.

[8] P. J. Olver, Equivalence, Invariant and Symmetry, Cambridge University Press, 1995.

[9] P. J. Olver, Invariant Theory and Differential Equations, in: *Invariant Theory*, S. S. Koh. ed., Lectures Notes in Mathematics, Vol. 1278, Springer-Verlage, New York, 1987, 62-80.

[10] Ahdrei D. Polyanin, Linear Partial Differential Equations for Engineers and Scientists, CRS Press, Washington D. C., 2002.

[11] G. Webb, M. P. Sorensen, M. Brio, A. R. Zakharian and J. V Moloney, *Variational principles, lie point symmetries, and similarity solutions of the vector maxwell equations in non-linear optics*, Phisica D 191 (2003) 5180.