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Warped products and quasi-Einstein metrics

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Abstract. Warped products provide a rich class of physically significant geometric objects. Warped product construction is an important method to produce a new metric with a base manifold and a fibre. We construct compact base manifolds with a positive scalar curvature which do not admit any non-trivial quasi-Einstein warped product, and noncompact complete base manifolds which do not admit any non-trivial Ricci-flat quasi-Einstein warped product.

Keywords: Warped product, quasi-Einstein space.
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1. Introduction

Warped product construction is a construction in the class of Riemannian manifolds that generalizes direct product. Warped product is an important method to produce a new metric with a base manifold and a fibre. This construction was introduced in ([1]) where it was used to construct a variety of complete Riemannian manifolds with negative sectional curvature and it served to give examples of new Riemannian manifolds. Warped products have significant applications, in general relativity, in the studies related to solutions of Einsteins equations ([2],

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[3]). Besides general relativity, warped product structures have also generated interest in many areas of geometry, especially due to their role in construction of new examples with interesting curvature and symmetry properties cf ([4], [7], [9], [12]). In ([6]) it was introduced the notion of quasi-Einstein manifold, notion that was generalize in ([5]). This method has been used for the construction of Einstein metrics on non-compact complete manifolds and other important examples in relativity and differential geometry ([2], [1]).
In this paper, we construct compact base manifolds with a positive scalar curvature which do not admit any non-trivial quasi-Einstein warped product. For a noncompact complete Riemannian manifolds, we prove that if a base manifold has at most quadratic volume growth and non-positive total scalar curvature, then non-trivial Ricci-flat quasi-Einstein warped product cannot exist.

Definition 1.1. A non-flat Riemannian manifold $(M^n,\ g)$, $(dimM = n \geq 3)$ is a *quasi-Einstein manifold* if its Ricci tensor Ric_M satisfies the condition

$$Ric_M(X,Y) = ag(X,Y) + bA(X)A(Y)$$

and is not identically zero, where a, b are scalars, $b \neq 0$ and A is a non-zero 1-form such that $g(X,U) = A(X)$, $\forall\ X \in \chi(M)$, U being a unit vector field, where n is the dimension of the manifold M. Here a and b are called the *associated scalars*, A is called the associated 1-form and U is called the *generator* of the manifold. If $b = 0$, then the manifold reduces to an Einstein space.

Definition 1.2. A non-flat Riemannian manifold $(M^n,\ g)$, $(dimM = n \geq 3)$ is a *generalized quasi-Einstein manifold* if its Ricci tensor Ric_M satisfies the condition

$$Ric_M(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y)$$

and is not identically zero, where a, b, c are certain non-zero scalars and A, B are two non-zero 1-form such that

$$g(X,U) = A(X),\ g(X,V) = B(X),\ g(U,V) = 0, \forall\ X \in \chi(M),$$

i.e., U, V are orthogonal vector fields on M. Here a and b are called the *associated scalars*, A, B are called the associated 1-form and U, V are called the *generator* of the manifold.

2. Warped Products

Let $N = (N^n, g_N)$ and $F = (F^k, g_F)$ be two Riemannian manifolds and f be a positive smooth function on N, where n and k are the dimension of N and F respectively. Consider the product manifold $N \times F$ with its projections $\pi : N \times F \to N$ and $\sigma : N \times F \to F$. The warped product $M = N \times_f F$ is the product manifold $N \times F$ with the Riemannian structure such that

$$\| X \|^2 = \| \pi^*(X) \|^2 + f^2(\pi(p)) \| \sigma^*(X) \|^2,$$

for any vector field X on M, where $*$ denotes the pull back. Thus $g_M = g_N + f^2 g_F$ holds on M. Here, N is called the base of M, and F is the fibre. The function f is called the warping function of the warped product([12]). The warped product is called a simply Riemannian product if f is a constant function. The concept of warped products was introduced by Bishop and O'Neil([1]) to construct examples of Riemannian manifold with negative curvature. We denote by Ric_N, Ric_F, H^f the lifts to M of the Ricci curvatures of N and F, and the Hessian of f, respectively. We have the following proposition from ([10]), ([4]) respectively:

Proposition 2.1: The Ricci curvature Ric_M of the warped product $M = N \times_f F$ satisfies:

$$Ric_M(X,Y) = Ric_N(X,Y) - \frac{k}{f} H^f(X,Y), \qquad (2.1)$$

$$Ric_M(X,V) = 0, \qquad (2.2)$$

$$Ric_M(V,W) = Ric_F(V,W) - g(V,W) f^{\#}, \qquad (2.3)$$

where $f^{\#} = -\frac{\Delta f}{f} + \frac{k-1}{f^2}|\nabla f|^2$ for any vectors $X, Y \in \chi(M)$ and $V, W \in \chi(M)$, where for H^f and Δf denote the Hessian of f and the Laplacian of f given by $H^f = Tr(H^f)$.

Proposition 2.2: Let $M = N \times_f F$ be an warped product. Then the scalar curvature of M is given by

$$\tau_M = \tau_N + \frac{\tau_F}{f^2} + 2k\frac{\Delta f}{f} - k(k-1)\frac{|\nabla f|^2}{f^2}. \qquad (2.4)$$

3. Quasi-Einstein Warped Products

Let $M = N \times_f F$ be an warped product manifold with $f : N \longrightarrow (0, \infty)$, $f \in C^\infty(N)$ and the metric $g_M = g_N + f^2 g_F$ which is also a quasi-Einstein manifold, that means its Ricci tensor satisfies:

$$Ric_M(X,Y) = a g_M(X,Y) + bA(X)A(Y). \qquad (3.1)$$

Starting from the above formula we want to compute the Ricci tensors of N and F. For that we will consider the following two cases: when U is tangent to N and when U is tangent to F. In ([8]), Dam Dumitru have studied the following theorem and corollary:

Theorem 3.1. *Let $M = N \times_f F$ be an warped product which is also a quasi-Einstein manifold, that is its Ricci tensor satisfies* (3.1).

a) *When U is tangent to the base N the Ricci tensors of N and F satisfy the following equations:*

$$Ric_N(X,Y) = ag_N(X,Y) + \frac{k}{f}H^f(X,Y) + bg_N(X,U)g_N(Y,U) \quad (3.2)$$

$$Ric_F(X,Y) = g_F(X,Y)[-f\Delta f + (k-1)|\nabla f|^2 + af^2]. \quad (3.3)$$

b) *When U is tangent to the fibre F the Ricci tensors of N and F satisfy the following equations:*

$$Ric_N(X,Y) = ag_N(X,Y) + \frac{k}{f}H^f(X,Y), \quad (3.4)$$

$$Ric_F(X,Y) = g_F(X,Y)[-f\Delta f + (k-1)|\nabla f|^2 + af^2] + bf^4 g_F(X,U)g_F(Y,U). \quad (3.5)$$

Corollary 3.1. *Taking the traces of (3.1), (3.2) and (3.3), we get the following*:

a) *When U is tangent to the base N, the scalar curvature of $M = N \times_f F$ is given by*

$$\tau_M = (m+k)a + b, \quad (3.6)$$

$$\tau_N = ma + k\frac{\Delta f}{f} + b, \quad (3.7)$$

$$\tau_F = k[-f\Delta f + (k-1)|\nabla f|^2 + af^2]. \quad (3.8)$$

Similarly *taking the traces of (3.1), (3.4) and (3.5), we obtain*:
b) *When U is tangent to the fibre F, the scalar curvature of $M = N \times_f F$ is given by*

$$\tau_M = (m+k)a + b, \quad (3.9)$$

$$\tau_N = ma + k\frac{\Delta f}{f}, \quad (3.10)$$

$$\tau_F = k[-f\Delta f + (k-1)|\nabla f|^2 + af^2] + bf^4. \quad (3.11)$$

4. Main Results of Quasi-Einstein Warped Products

First, we observe that there are infinitely many metrics on a base manifold N of dimension $m \geq 3$ such that $M = N \times_f F$ cannot be a non-trivial quasi-Einstein warped product.A differential manifold of dimension $m \geq 3$ admits infinitely many different conformal classes. Therefore, there are infinitely many metrics on a base manifold N of dimension $m \geq 3$ such that there are no non-trivial Einstein warped products on N. From [8], we know the existence of quasi-Einstein warped

products. D. Dumitru ([8]), considered two cases:
Case 1: When U is tangent to F. Let $M = N \times_f F$ be an warped product with compact and connected base and $dimN = m \geq 1$ and $dimF = k \geq 1$ which is also a quasi-Einstein manifold with $Ric_M(X,Y) = ag_M(X,Y) + bA(X)A(Y)$, $a, b \neq 0$, $A(X) = g_M(X,U)$ for every $X, Y \in \chi(M)$ with U an unitary vector field tangent to F. Then we have
i) If $m = 1$ or $k = 1$, then M is a simply Riemannian product.
ii) If $dimN = m \geq 2$, $dimF = k \geq 2$ and $b \neq 0$, then M reduces to a simply Riemannian product.

Case 2: When U is tangent to N. Let $M = N \times_f F$ be an warped product with N compact and connected, $dimN = m \geq 1$, $dimF = k \geq 1$ which is also a quasi-Einstein manifold with $Ric_M(X,Y) = ag_M(X,Y) + bA(X)A(Y)$, $a, b \in R$, $A(X) = g_M(X,U)$ for every $X, Y \in \chi(M)$ with U an unitary vector field tangent to F. Then
i) If $m = 1$ or $k = 1$, then M is a simply Riemannian product.
ii) If $a \leq 0$ then M reduces to a simply Riemannian product.
iii) If F has negative scalar curvature, then M reduces to a simply Riemannian product.
Now in our paper, we consider U is tangent to N only when $a > 0$ and $b > 0$. Then we have the following theorem:
Theorem 4.1. *Let (N_1^r, g_1), (N_2^m, g_2) be compact Riemannian manifolds with scalar curvature τ_{N_1} and τ_{N_2}, respectively. Assume that τ_{N_1} is a positive constant and (N_2^m, g_2) has a non-positive total scalar curvature, i.e., $\int_{N_2} \tau_{N_2} dV_{g_2} \leq 0$ with $|\tau_{N_2}| < \tau_{N_1}$. Consider $N = N_1 \times N_2$ with the product metric $g_N = g_1 + g_2$ and positive scalar curvature τ_N. Then there are no non-trivial quasi-Einstein warped products $(M, g) = N \times_f F$ with base (N, g_N).*

Proof. Let $f = f(x_1, x_2, ..., x_r, x_{r+1} ..., x_{r+m})$ be a positive smooth function on (N_1^r, N_2^m). By taking the trace of (3.2) for $i = 1$ *to* $i = r$, and $i = r + 1$ *to* $i = r + m$ respectively, we have

$$ra = \tau_{N_1} - k\frac{\Delta_{g_1} f}{f} - b, \qquad (4.1)$$

$$ma = \tau_{N_2} - k\frac{\Delta_{g_2} f}{f} - b. \qquad (4.2)$$

Since τ_{N_1} is constant and $\Delta_{g_1} f$ should have a fixed sign in (4.1), which implies that f does not depend on $x_1, ..., x_r$. Integrating (4.2) on N_2, we have

$$\begin{aligned}\int_{N_2} \tau_{N_2} dV_{g_2} &= \int_{N_2} ma + k\frac{\Delta_{g_1} f}{f} + b\ dV_{g_2} \\ &= \int_{N_2} ma + k|\frac{\nabla f}{f}|^2_{g_2} + b\ dV_{g_2} \qquad (4.3)\end{aligned}$$

which is not possible since $a > 0$ and $b > 0$ and $\int_{N_2} \tau_{N_2} dV_{g_2} \leq 0$. So, we conclude that f is a constant function.

Theorem 4.2. *Let (N_1^r, g_1) a compact manifold of dimension $r \geq 3$ with constant scalar curvature τ_{N_1} and (N_k, g_k) be two-dimensional compact Riemannian manifolds for $k = 2, ..., m$ and denoted by $(N, g_N) = (\Pi_{k=1}^{k=m} N_k, \Sigma_{k=1}^m g_k)$. If $N \times_f F$ is quasi-Einstein, then the warped product should be trivial.*

Proof. By taking the trace of (3.2) on N_1,..., N_m respectively, we have

$$ra = \tau_{N_1} - k\frac{\Delta_{g_1} f}{f} - b, \qquad (4.4)$$

$$2a = \tau_{N_k} - k\frac{\Delta_{g_k} f}{f} - b, for\ k = 2, ..., m. \qquad (4.5)$$

In [11], there is a proof of non-existence non-trivial quasi-Einstein warped product over a compact two-dimensional base manifold. Therefore, there is no nonconstant function on each N_k satisfying (4.5) for $k = 2, ..., m$ which implies that f is a function of N_1 only. Since f satisfies (4.4) and τ_{N_1} is a constant, f should be a constant by the argument in the above of theorem 4.3. So, our theorem is proved.

Next we construct a noncompact complete base manifold (N, g_N) such that there is no non-trivial Ricci-flat quasi-Einstein warped product $M = N \times_f F$ with base (N, g_N). For this, we introduce some notations. Let (N, g_N) be a noncompact complete Riemannian manifold. For $\Omega \subset N$ we let $|\Omega|$ be the volume of Ω with respect to the metric g_N. $B(R) = \{x \in N | dist(p, x) \leq R\}$ for a fixed point $p \in N$. (N, g_N) has at most quadratic volume growth if there exists a constant c such that and $\limsup_{R\to\infty} \frac{|B(R)|}{R^2} \leq c$ For the construction of (N, g_N), we estimate the solutions of (3.7).

Theorem 4.3. *Let (N, g_N) be a noncompact complete Riemannian manifold with scalar curvature τ_N. Assume that (N, g_N) satisfies the following conditions:*

(a) (N, g_N) has at most quadratic volume growth.

b)$-\infty \leq \int_N \tau_N dV_{g_n} \leq 0$ and $\int_N \tau_N^+ dV_{g_n}$ is finite, where $\tau_N^+(x) = \max(0, \tau_N^+(x))$.

Then the followings hold:

i) If $(N \times F, g_N + f^2 g_F)$ is Ricci-flat quasi-Einstein, then f should be a constant.

ii) If there exists a point q in N such that $\tau_N(q) < 0$, then $(N \times F, g_N + f^2 g_F)$ cannot be Ricci-flat quasi-Einstein.

Proof. Choose a smooth function ϕ^2 such that $0 \leq \phi \leq 1$ on N, $\phi = 1$ on $B(R)$, $\phi = 0$ outside of $B(2R)$ and $|\nabla\phi| \leq \frac{c}{R}$ where c is a constant. From the above condition (b), for any given small ϵ there exists a sufficiently large R_0 such that $\int_{N-B(R)} \tau_N^+ dV_g \leq d\epsilon$ and $\int_{B(R)} \tau_N dV_g < d\epsilon$ for $R \geq R_0$. This implies

$$\int_{B(2R)} \tau_N \phi^2 dV_{g_N} \leq \int_{B(R)} \tau_N dV_{g_N} + \int_{B(2R)-B(R)} \tau_N^+ dV_{g_N} < 2d\epsilon. \quad (4.6)$$

Multiplying ϕ^2 on (3.7) with $a = 0$,

$$\begin{aligned} \int_N \phi^2 |\frac{\nabla f}{f}|^2_{g_N} &- \frac{\phi^2}{k}\tau_N dV_{g_N} + \frac{b}{k}\int_N \phi^2 dV_{g_N} \\ &= \int_N 2\phi\nabla\phi\frac{\nabla f}{f} dV_{g_N} + \frac{b}{k}\int_N \phi^2 dV_{g_N} \end{aligned}$$

$$\leq \int_N \frac{\phi^2}{2}|\frac{\nabla f}{f}|^2_{g_N} + 2|\nabla\phi|^2 dV_{g_N} + \frac{b}{k}\int_N \phi^2 dV_{g_N}, \quad (4.7)$$

where the Hölder inequality is used in (4.8). Therefore,

$$\begin{aligned} \int_N \frac{\phi^2}{2}|\frac{\nabla f}{f}|^2_{g_N} dV_{g_N} + \frac{b}{k}\int_N \phi^2 dV_{g_N} &\leq \int_N 2|\nabla\phi|^2 + \frac{\phi^2}{k}\tau_N dV_{g_N} \\ &+ \frac{b}{k}\int_N \phi^2 dV_{g_N}. \quad (4.8) \end{aligned}$$

So, from (4.9), we have

$$\begin{aligned} \int_N \frac{\phi^2}{2k}[k|\frac{\nabla f}{f}|^2_{g_N} + 2b] dV_{g_N} &\leq \int_N 2|\nabla\phi|^2 + \frac{\phi^2}{k}(\tau_N + b) dV_{g_N} \\ &\leq \frac{2c^2}{R^2}|B(2R) - B(R)| + 2\epsilon \\ &\leq c'. \quad (4.9) \end{aligned}$$

where c' is some positive constant and the quadratic volume growth condition is used in (4.10). Therefore, the integral $\int_N [k|\frac{\nabla f}{f}|^2_{g_N} - 2b] dV_{g_N}$ is uniformly bounded. From (4.7), we have

$$\begin{aligned} \int_N \frac{\phi^2}{2k}[k|\frac{\nabla f}{f}|^2_{g_N} + 2b] dV_{g_N} &\leq \frac{2c}{R}\int_{B(2R)-B(R)} |\frac{\nabla f}{f}|_{g_N} dV_{g_N} + \int_N \frac{\phi^2}{k}\tau_N dV_{g_N} \\ &\leq 2c\{\int_{B(2R)-B(R)} |\frac{\nabla f}{f}|^2_{g_N}\}^{\frac{1}{2}} \frac{|B(2R) - B(R)|^{\frac{1}{2}}}{R} \\ &+ 2\epsilon \quad (4.10) \end{aligned}$$

By the condition (a) and $\int_N \frac{\phi^2}{2k}[k|\frac{\nabla f}{f}|^2_{g_N}+2b]dV_{g_N} < \infty$, $\int_{B(R)} |\frac{\nabla f}{f}|^2_{g_N} dV_{g_N}$ goes to zero as $R \to \infty$. We conclude that f should be a constant. Furthermore, if there exists a point $q \in N$ with $S_{N(q)} < 0$, then the constant f does not satisfy (3.7).

Note: One can also find out warped product on generalized quasi-Einstein manifold and on other Riemannian manifolds in a similar way of the present paper.

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