Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 1735-0611

CJMS. **5**(2)(2016), 98-105

Some Results for the Jacobi-Dunkl Transform in the Space $L^p(\mathbb{R}, A_{\alpha,\beta}(x)dx)$

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Abstract. In this paper, using a generalized Jacobi-Dunkl translation operator, we obtain a generalization of Titchmarsh's theorem for the Dunkl transform for functions satisfying the (ϕ, p)-Lipschitz Jacobi-Dunkl condition in the space $L^p(\mathbb{R}, A_{\alpha,\beta}(x)dx)$, $\alpha \geq \beta \geq \frac{-1}{2}$, $\alpha \neq \frac{-1}{2}$.

Keywords: Jacobi-Dunkl operator, Jacobi-Dunkl transform, generalized Jacobi-Dunkl translation.

2000 Mathematics subject classification: 42B37.

1. Introduction and Preliminaries

Titchmarsh's ([8], Theorem 85) characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy-Lipschitz Condition by means of an asymptotic estimate growth of the norm of their Fourier transform, namely we have:

Theorem 1.1. [8] *Let $\alpha \in (0,1)$ and assume that $f \in L^2(\mathbb{R})$. Then the following are equivalents:*
(a) $\quad \|f(x+h) - f(x)\| = O(h^\alpha), \quad$ *as* $\quad h \to 0,$

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Received: 1 July 2015
Revised: 24 July 2016
Accepted: 6 August 2016

(b) $\displaystyle\int_{|\lambda|\geq r} |\widehat{f}(\lambda)|^2 d\lambda = O(r^{-2\alpha}), \quad as \quad r \to \infty,$

where $\widehat{f}$ stands for the Fourier transform of f.

A similar result of Theorem 1.1 has been established for the Jacobi transform in the space $L^2(\mathbb{R}, A_{\alpha,\beta}(x)dx)$ (see [11]). In this paper, we prove a generalization of Theorem 1.1 for the Jacobi-Dunkl transform for functions satisfying the (ϕ, p)-Lipschitz Jacobi-Dunkl condition in the space $L^p(\mathbb{R}, A_{\alpha,\beta}(x)dx), 1 < p \leq 2$. For this purpose, we use the generalized Jacobi-Dunkl translation operator.

In this section, we recapitulate from ([1]-[6]) some results related to the harmonic analysis associated with Jacobi-Dunkl operator $\Lambda_{\alpha,\beta}$.
The Jacobi-Dunkl function with parameters $(\alpha,\beta), \alpha \geq \beta \geq \frac{-1}{2}, \alpha \neq \frac{-1}{2}$, is defined by the formula:

$$\forall x \in \mathbb{R}, \psi_\lambda^{\alpha,\beta}(x) = \begin{cases} \varphi_\mu^{\alpha,\beta}(x) - \frac{i}{\lambda}\frac{d}{dx}\varphi_\mu^{\alpha,\beta}(x), & \text{if } \lambda \in \mathbb{C}\backslash\{0\}, \\ 1, & \text{if } \lambda = 0, \end{cases}$$

with $\lambda^2 = \mu^2 + \rho^2$, $\rho = \alpha + \beta + 1$ and $\varphi_\mu^{\alpha,\beta}$ is the Jacobi function given by:

$$\varphi_\mu^{\alpha,\beta}(x) = F\left(\frac{\rho + i\mu}{2}, \frac{\rho - i\mu}{2}, \alpha + 1, -(\sinh(x))^2\right),$$

F is the Gauss hypergeometric function (see [1],[7]).
$\psi_\lambda^{\alpha,\beta}$ is the unique C^∞-solution on $\mathbb{R}$ of the differential-difference equation

$$\begin{cases} \Lambda_{\alpha,\beta}\mathcal{U} = i\lambda\mathcal{U}, & \lambda \in \mathbb{C}, \\ \mathcal{U}(0) = 1, \end{cases}$$

(see [5]), where $\Lambda_{\alpha,\beta}$ is the Jacobi-Dunkl operator given by:

$$\Lambda_{\alpha,\beta}\mathcal{U}(x) = \frac{d\mathcal{U}(x)}{dx} + \frac{A'_{\alpha,\beta}(x)}{A_{\alpha,\beta}(x)} \times \left(\frac{\mathcal{U}(x) - \mathcal{U}(-x)}{2}\right),$$

with

$$A_{\alpha,\beta}(x) = 2^\rho(\sinh|x|)^{2\alpha+1}(\cosh|x|)^{2\beta+1},$$

i.e.,

$$\Lambda_{\alpha,\beta}\mathcal{U}(x) = \frac{d\mathcal{U}(x)}{dx} + [(2\alpha+1)\coth x + (2\beta+1)\tanh x] \times \frac{\mathcal{U}(x) - \mathcal{U}(-x)}{2}.$$

Using the relation

$$\frac{d}{dx}\varphi_\mu^{\alpha,\beta}(x) = -\frac{\mu^2 + \rho^2}{4(\alpha+1)}\sinh(2x)\varphi_\mu^{\alpha+1,\beta+1}(x),$$

the function $\psi_\lambda^{\alpha,\beta}$ can be written in the form above (see [2])

$$\psi_\lambda^{\alpha,\beta}(x)=\varphi_\mu^{\alpha,\beta}(x)+i\frac{\lambda}{4(\alpha+1)}sinh(2x)\varphi_\mu^{\alpha+1,\beta+1}(x),\quad x\in\mathbb{R},$$

where $\lambda^2=\mu^2+\rho^2,\quad \rho=\alpha+\beta+1.$
Denote $L^p_{\alpha,\beta}(\mathbb{R})=L^p_{\alpha,\beta}(\mathbb{R},A_{\alpha,\beta}(x)dx), 1<p\leq 2$ the space of measurable functions f on $\mathbb{R}$ such that

$$\|f\|_{p,\alpha,\beta}=\left(\int_{\mathbb{R}}|f(x)|^pA_{\alpha,\beta}(x)dx\right)^{1/p}<+\infty.$$

Using the eigenfunctions $\psi_\lambda^{\alpha,\beta}$ of the operator $\Lambda_{\alpha,\beta}$ called the Jacobi-Dunkl kernels, we define the Jacobi-Dunkl transform by

$$\mathcal{F}_{\alpha,\beta}f(\lambda)=\int_{\mathbb{R}}f(x)\psi_\lambda^{\alpha,\beta}(x)A_{\alpha,\beta}(x)dx,\quad \lambda\in\mathbb{R},$$

and the inversion formula by

$$f(t)=\int_{\mathbb{R}}\mathcal{F}_{\alpha,\beta}f(\lambda)\psi_{-\lambda}^{\alpha,\beta}(t)d\sigma(\lambda),$$

where

$$d\sigma(\lambda)=\frac{|\lambda|}{8\pi\sqrt{\lambda^2-\rho^2}|C_{\alpha,\beta}(\sqrt{\lambda^2-\rho^2})|}\mathbb{I}_{\mathbb{R}\backslash]-\rho,\rho[}(\lambda)d\lambda.$$

Here,

$$C_{\alpha,\beta}(\mu)=\frac{2^{\rho-i\mu}\Gamma(\alpha+1)\Gamma(i\mu)}{\Gamma(\frac{1}{2}(\rho+i\mu))\Gamma(\frac{1}{2}(\alpha-\beta+1+i\mu))},\quad \mu\in\mathbb{C}\backslash(i\mathbb{N}),$$

and $\mathbb{I}_{\mathbb{R}\backslash]-\rho,\rho[}$ is the characteristic function of $\mathbb{R}\backslash]-\rho,\rho[$.
The Jacobi-Dunkl transform is a unitary isomorphism from $L^2_{\alpha,\beta}(\mathbb{R})$ onto $L^2(\mathbb{R},d\sigma(\lambda))$, i.e.,

$$\|f\|_{2,\alpha,\beta}=\|\mathcal{F}_{\alpha,\beta}(f)\|_{L^2(\mathbb{R},d\sigma(\lambda))}. \tag{1.1}$$

Plancherel's theorem (1.1) and the Marcinkiewics interpolation theorem (see [8]) we get for $f\in L^p_{\alpha,\beta}(\mathbb{R})$ with $1<p\leq 2$ and q such that $\frac{1}{p}+\frac{1}{q}=1$,

$$\|\mathcal{F}_{\alpha,\beta}(f)\|_{L^q(\mathbb{R},d\sigma(\lambda))}\leq K\|f\|_{p,\alpha,\beta}, \tag{1.2}$$

where K is a positive constant (see [6]).
The operator of Jacobi-Dunkl translation is defined by :

$$T_xf(y)=\int_{\mathbb{R}}f(z)d\nu_{x,y}^{\alpha,\beta}(z),\quad \forall x,y\in\mathbb{R},$$

where $\nu_{x,y}^{\alpha,\beta}(z), \quad x,y\in\mathbb{R}$ are the signed measures given by

$$d\nu_{x,y}^{\alpha,\beta}(z)=\begin{cases} K_{\alpha,\beta}(x,y,z)A_{\alpha,\beta}(z)dz, & \text{if } x,y\in\mathbb{R}^*,\\ \\ \delta_x, & \text{if } y=0,\\ \delta_y, & \text{if } x=0.\end{cases}$$

Here, δ_x is the Dirac measure at x. And,

$$\begin{aligned} K_{\alpha,\beta}(x,y,z) &= M_{\alpha,\beta}(\sinh(|x|)\sinh(|y|)\sinh(|z|))^{-2\alpha}\mathbb{I}_{I_{x,y}}\times\int_0^\pi \rho_\theta(x,y,z)\\ &\times \quad (g_\theta(x,y,z))_+^{\alpha-\beta-1}\sin^{2\beta}\theta d\theta,\end{aligned}$$

where

$$\begin{aligned} I_{x,y} &= [-|x|-|y|,-||x|-|y||]\cup[||x|-|y||,|x|+|y|]\\ \rho_\theta(x,y,z) &= 1-\sigma_{x,y,z}^\theta+\sigma_{z,x,y}^\theta+\sigma_{z,y,x}^\theta,\end{aligned}$$

$$\forall z\in\mathbb{R},\theta\in[0,\pi],\sigma_{x,y,z}^\theta=\begin{cases}\frac{\cosh(x)+\cosh(y)-\cosh(z)\cos(\theta)}{\sinh(x)\sinh(y)}, & \text{if } xy\neq 0,\\ \\ 0, & \text{if } xy=0,\end{cases}$$

$$g_\theta(x,y,z)=1-\cosh^2(x)-\cosh^2(y)-\cosh^2(z)+2\cosh(x)\cosh(y)\cosh(z)\cos\theta,$$

$$t_+=\begin{cases} t, & \text{if } t>0,\\ \\ 0, & \text{if } t\le 0,\end{cases}$$

and,

$$M_{\alpha,\beta}=\begin{cases}\frac{2^{-2\rho}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})}, & \text{if } \alpha>\beta,\\ \\ 0, & \text{if } \alpha=\beta.\end{cases}$$

In [2], we have

$$\mathcal{F}_{\alpha,\beta}(T_hf)(\lambda) = \psi_\lambda^{\alpha,\beta}(h)\mathcal{F}_{\alpha,\beta}(f)(\lambda), \tag{1.3}$$

$$\mathcal{F}_{\alpha,\beta}(\Lambda_{\alpha,\beta}f)(\lambda) = i\lambda\mathcal{F}_{\alpha,\beta}(f)(\lambda). \tag{1.4}$$

For $\alpha\ge\frac{-1}{2}$, we introduce the normalized Bessel function of first kind and order α [10] defined by:

$$j_\alpha(x)=\Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n(x/2)^{2n}}{n!\Gamma(n+\alpha+1)}, \quad x\in\mathbb{R}.$$

Moreover, we see that

$$\lim_{x\to 0}\frac{j_\alpha(x)-1}{x^2}\neq 0,$$

by consequence, there exists $C_1>0$ and $\eta>0$ satisfying

$$|x|\le\eta\Rightarrow|j_\alpha(x)-1|\ge C_1|x|^2. \tag{1.5}$$

Lemma 1.2. *Let $\alpha \geq \beta \geq \frac{-1}{2}, \alpha \neq \frac{-1}{2}$. Then for $|\nu| \leq \rho$, there exists a positive constant C_2 such that*

$$|1 - \varphi^{\alpha,\beta}_{\mu+i\nu}(x)| \geq C_2|1 - j_\alpha(\mu x)|.$$

Proof. (See [4], Lemma 9). □

Denote by $L^m_p(\Lambda_{\alpha,\beta}), 1 < p \leq 2, m = 0, 1, 2...$, the class of functions $f \in L^p_{\alpha,\beta}(\mathbb{R})$ that have on $\mathbb{R}$ generalized derivatives $f'(x), f''(x), ..., f^{(2m)}(x)$ in the sense of Levi (see [9]) and belong to $L^p_{\alpha,\beta}(\mathbb{R})$ with $\Lambda^m_{\alpha,\beta} f \in L^p_{\alpha,\beta}(\mathbb{R})$. i.e.,

$$L^m_p(\Lambda_{\alpha,\beta}) = \left\{ f \in L^p_{\alpha,\beta}(\mathbb{R}) / \Lambda^m_{\alpha,\beta} f \in L^p_{\alpha,\beta}(\mathbb{R}) \right\},$$

where $\Lambda^0_{\alpha,\beta} f = f$, $\Lambda^m_{\alpha,\beta} f = \Lambda_{\alpha,\beta}(\Lambda^{m-1}_{\alpha,\beta} f), m = 0, 1, 2....$

2. Main result

In this section we give the main result of this paper. We need first to define (ϕ, p)-Lipschitz Jacobi-Dunkl class.
Denote N_h by

$$N_h = T_h + T_{-h} - 2I,$$

where I is the unit operator in the space $L^p_{\alpha,\beta}(\mathbb{R})$.

Definition 2.1. A function $f \in L^m_p(\Lambda_{\alpha,\beta})$ is said to be in (ϕ, p)-Lipschitz Jacobi-Dunkl class, denoted by $Lip(\phi, p, \alpha, \beta)$, if

$$\|N_h \Lambda^m_{\alpha,\beta} f\|_{p,\alpha,\beta} = O(\phi(h)), \quad as \quad h \to 0,$$

where $m = 0, 1, 2, ...$ and ϕ is a continuous increasing function on $[0, \infty)$, satisfying $\phi(0) = 0$ and $\phi(ts) = \phi(t)\phi(s)$ for all $t, s \in [0, \infty)$.

Lemma 2.2. *For $f \in L^m_p(\Lambda_{\alpha,\beta})$, then*

$$\left(\int_{\mathbb{R}} 2^q \lambda^{qm} |\varphi^{\alpha,\beta}_\mu(h) - 1|^q |\mathcal{F}_{\alpha,\beta} f(\lambda)|^q d\sigma(\lambda) \right)^{\frac{1}{q}} \leq K \|N_h \Lambda^m_{\alpha,\beta} f\|_{p,\alpha,\beta},$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $m = 0, 1, 2, ...$

Proof. From (1.4), we have

$$\mathcal{F}_{\alpha,\beta}(\Lambda^m_{\alpha,\beta} f)(\lambda) = i^m \lambda^m \mathcal{F}_{\alpha,\beta}(f)(\lambda), \quad m = 0, 1, 2, ... \tag{2.1}$$

We use formulas (1.3) and (2.1), we conclude that

$$\mathcal{F}_{\alpha,\beta}(N_h \Lambda^m_{\alpha,\beta} f)(\lambda) = i^m(\psi^{(\alpha,\beta)}_\lambda(h) + \psi^{(\alpha,\beta)}_\lambda(-h) - 2)\lambda^m \mathcal{F}_{\alpha,\beta}(f)(\lambda),$$

Since

$$\psi^{(\alpha,\beta)}_\lambda(h) = \varphi^{\alpha,\beta}_\mu(h) + i\frac{\lambda}{4(\alpha+1)} \sinh(2h) \varphi^{\alpha+1,\beta+1}_\mu(h),$$

$$\psi_\lambda^{(\alpha,\beta)}(-h) = \varphi_\mu^{\alpha,\beta}(-h) - i\frac{\lambda}{4(\alpha+1)}\sinh(2h)\varphi_\mu^{\alpha+1,\beta+1}(-h),$$

and $\varphi_\mu^{\alpha,\beta}$ is even, then

$$\mathcal{F}_{\alpha,\beta}(N_h\Lambda_{\alpha,\beta}^m f)(\lambda) = 2i^m(\varphi_\mu^{\alpha,\beta}(h) - 1)\lambda^m \mathcal{F}_{\alpha,\beta}(f)(\lambda).$$

By formula 1.2, we have the result. □

Theorem 2.3. *Let f belong to $Lip(\phi, p, \alpha, \beta)$. Then*

$$\int_{|\lambda|\geq r} \lambda^{qm}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) = O(\phi(r^{-q})), \quad as \quad r\to\infty,$$

where $\frac{1}{p}+\frac{1}{q} = 1$ and $m = 0, 1, 2, ...$

Proof. Assume that $f \in Lip(\phi, p, \alpha, \beta)$, then we have

$$\|N_h\Lambda_{\alpha,\beta}^m f\|_{p,\alpha,\beta} = O(\phi(h)), \quad as \quad h \to 0.$$

From Lemma 2.2, we have

$$\int_{\mathbb{R}} \lambda^{qm}|\varphi_\mu^{\alpha,\beta}(h) - 1|^q|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda) \leq \frac{K^q}{2^q}\|N_h\Lambda_{\alpha,\beta}^m f\|_{p,\alpha,\beta}^q.$$

By (1.5) and Lemma 1.2, we get:

$$\int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}} \lambda^{qm}|1-\varphi_\mu^{\alpha,\beta}(h)|^q|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \geq C_1^q C_2^q \int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}} |\mu h|^{2q}\lambda^{qm}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda).$$

From $\frac{\eta}{2h} \leq |\lambda| \leq \frac{\eta}{h}$ we have

$$\begin{aligned}\left(\frac{\eta}{2h}\right)^2 \quad - \quad & \rho^2 \leq \mu^2 \leq \left(\frac{\eta}{h}\right)^2 - \rho^2 \\ \Rightarrow \quad & \mu^2 h^2 \geq \frac{\eta^2}{4} - \rho^2 h^2.\end{aligned}$$

Take $h \leq \frac{\eta}{3\rho}$, then we have $\mu^2h^2 \geq C_3 = C_3(\eta)$.
So,

$$\int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}} \lambda^{qm}|1-\varphi_\mu^{\alpha,\beta}(h)|^q|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \geq C_1^q C_2^q C_3^q \int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}} \lambda^{qm}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda).$$

There exists then a positives constants C and K_1 such that

$$\begin{aligned}\int_{\frac{\eta}{2h}\leq|\lambda|\leq\frac{\eta}{h}} \lambda^{qm}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \quad &\leq \quad C\int_{\mathbb{R}} \lambda^{qm}|1-\varphi_\mu^{\alpha,\beta}(h)|^q|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \\ &\leq \quad K_1\phi^q(h) = K_1\phi(h^q).\end{aligned}$$

For all $0 < h < \frac{\eta}{3\rho}$. Then we have,

$$\int_{r\leq|\lambda|\leq 2r} \lambda^{qm}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \leq K_2\phi(r^{-q}), \quad r\to\infty.$$

where $K_2 = K_1\phi(\eta^q 2^{-q})$.
Furthermore, we obtain

$$\begin{aligned}\int_{|\lambda|\geq r}\lambda^{qm}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) &= \left(\int_{r\leq|\lambda|\leq 2r}+\int_{2r\leq|\lambda|\leq 4r}+\int_{4r\leq|\lambda|\leq 8r}+\cdots\right)\lambda^{qm}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda)\\ &\leq K_2\phi(r^{-q})+K_2\phi((2r)^{-q})+K_2\phi((4r)^{-q})+\cdots\\ &\leq K_2\phi(r^{-q})+K_2\phi(2^{-q})\phi(r^{-q})+K_2\phi((2^{-q})^2)\phi(r^{-q})+\cdots\\ &\leq K_2\phi(r^{-q})(1+\phi(2^{-q})+\phi((2^{-q})^2)+\cdots).\end{aligned}$$

We have $\phi(2^{-q}) < 1$, then

$$\int_{|\lambda|\geq r}\lambda^{qm}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda)\leq K_3\phi(r^{-q}),$$

where $K_3 = K_2(1-\phi(2^{-q}))^{-1}$.
Finally, we get

$$\int_{|\lambda|\geq r}\lambda^{qm}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) = O(\phi(r^{-q})),\quad as\quad r\to\infty.$$

Thus, the proof is finished. □

Corollary 2.4. *Let $f\in L_p^m(\Lambda_{\alpha,\beta})$, and let*

$$\|N_h\Lambda_{\alpha,\beta}^m f\|_{p,\alpha,\beta} = O(\phi(h)),\quad as\quad h\to 0.$$

Then

$$\int_{|\lambda|\geq r}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) = O(r^{-qm}\phi(r^{-q})),\quad as\quad r\to\infty,$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $m = 0,1,2,....$

Acknowledgements
The authors would like to thank the referee for his valuable comments and suggestions.

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