Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 1735-0611

CJMS. **5**(2)(2016), 47-54

Carter–Penrose diagrams and differential spaces

Krzysztof Drachal
Faculty of Mathematics and Information Science
Warsaw University of Technology, Poland[1]

Abstract. In this paper it is argued that a Carter–Penrose diagram can be viewed as a differential space.

Keywords: Carter–Penrose diagram, Conformal diagram, Differential spaces.

2000 Mathematics subject classification: 58A40, 83C57, 83C75.

1. Introduction

Carter–Penrose diagrams are a very useful method to depict a spacetime in General Relativity. Sometimes they are called just Penrose diagrams or conformal diagrams [5, 26, 11, 18, 12, 2]. We will show that they can be viewed as Sikorski differential spaces [22, 23, 25, 7, 4]. Such a hypothesis is given in [9].

We will briefly remind the construction of a Penrose diagram. Let M be a 4–dimensional manifold and g be a Lorentzian metric. Then, the pair (M, g) is a spacetime [5, 26, 11, 18, 12, 21].

We assume that the considered spacetime M is spherically symmetric [20, 16].

Definition 1.1. If the group of isometries G of a spacetime M contains a subgroup, which is isomorphic to the rotation group $SO(3)$ and, moreover, orbits of the group G are 2–dimensional spheres, then the spacetime M will be called spherically symmetric.

[1] Corresponding author: k.drachal@mini.pw.edu.pl
Received: 1 December 2015
Revised: 28 January 2016
Accepted: 4 February 2016

Roughly speaking, the above definition means that for a spherically symmetric spacetime rotations do not change the metric. Indeed, a Killing vector field is such a vector field V for which $\mathcal{L}_V g = 0$ holds, where $\mathcal{L}$ is the Lie derivative. For example, the well–known Birkhoff–Jebsen theorem [1, 14] states that any spherically symmetric solution of the vacuum Einstein field equations is static and asymptotically flat. As a consequence, the exterior solution is just the Schwarzschild metric.

Assumption of a spherical symmetry is important, because every point on the Penrose diagram corresponds to a 2–dimensional sphere. For example, $\widetilde{M} \ni p \mapsto S^2_p$. The construction of a Penrose diagram is done in accordance with the below principles.

First, the null geodesics are drawn as 45° lines. Any two such lines intersect on the Penrose diagram, if and only if they correspond to null geodesics which intersect on the initially considered spacetime. Secondly, a spacetime is drawn as a half–plane. Then, the spacetime is conformally rescalled (from M to $\widetilde{M}$). Notice, that a conformal rescalling does not changes the angles. Roughly speaking, the conformal rescalling can be described as the following change of the metric $g \mapsto \Omega^2 g$, or of the line element $ds^2 \mapsto \Omega ds^2$, where the function Ω is such that $\Omega \sim O(r^{-1})$ and it is non–zero everywhere.

Because we restricted to spherically symmetric spacetimes, it is enough to consider only two directions on the diagram, i.e., the spatial r and temporal t. Remember, that the space is "the same" in all directions due to the assumptions.

For example, the Penrose diagram of the Minkowski spacetime is sketched on Figure 1. The dashed lines are the paths of photons, i.e., null geodesics. The bowed line is the example of a timelike geodesic, i.e., a path of a massive particle.

It is easy to notice, that the diagram can be simplified, i.e., like on the Figure 2, because we can imagine that the rotation is done around the t axis. (Remember, the assumption of a spherical symmetry.)

On the Fig. 3 B represents the black hole region of the spacetime. The bowed line is a word line of a matter collapsing to the black hole. The event horizon is represented by the dashed line.

2. Main result

Let M be a set such that $M \neq \emptyset$ and let C be a collection of some real functions on M. M can be made a topological space (M, τ). In particular, τ is the weakest topology such that every function from C is continuous.

Consider the following operations:

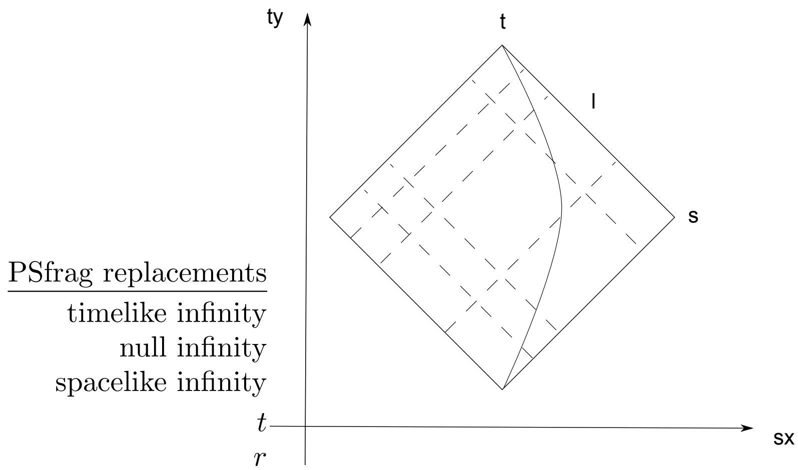

FIGURE 1. Penrose diagram of the Minkowski spacetime

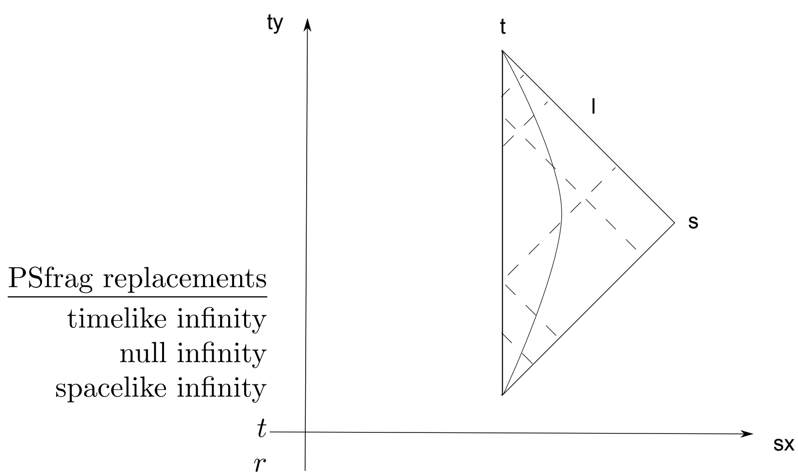

FIGURE 2. Penrose diagram of the Minkowski spacetime

(1): Let f be a real function defined on a subset N of M. If for all $x \in N$ there exists an open set $U \in \tau$ such that there exists a function $g \in C$ such that $f|_U = g|_U$, then f will be called a local C–function. And the collection of all local C–functions on N will be denoted by C_N.

(2): Moreover, we define compositions of functions from C with smooth n–variable real functions for every $n \in \mathbb{N}$, i.e.,

$$\mathrm{sc}C := \{H \circ (f_1, \dots, f_n) \mid n \in \mathbb{N}\ ,\ H \in C^\infty(\mathbb{R}^n)\ ,\ f_1, \dots, f_n \in C\} \quad .$$

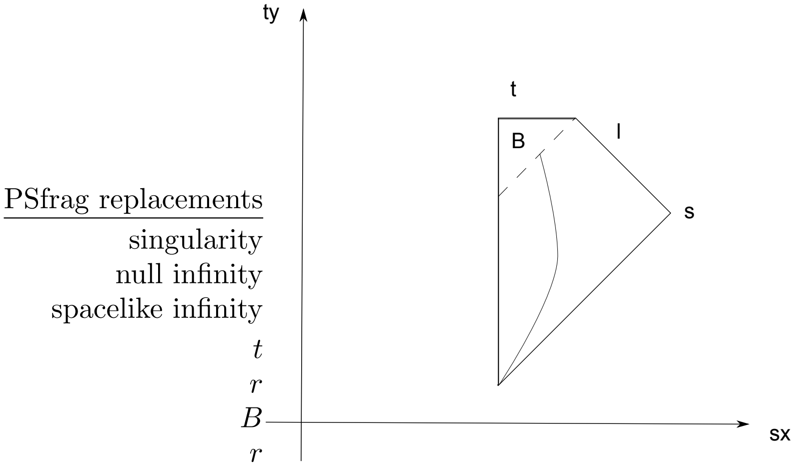

FIGURE 3. Penrose diagram of the black hole

Definition 2.1. If C denotes some collection of real functions on a non–empty set M and the structure C is closed with respect to both operations (1) and (2) above, i.e., $C = (\text{sc}C)_M$, then (M, C) will be called a differential space [22, 23, 25, 7].

We adapt the notation and symbols used in the Introduction.

Suppose that we study the spacetime (M, g). We can also understand it as a differential space $(M, C^\infty(M))$ equipped with the additional metric structure given by g [7].

Next, the conformal rescalling is done. The metric g is mapped to $\widetilde{g} := \Omega g$. As a result, we have the spacetime $(\widetilde{M}, \widetilde{g})$, or a differential space $(\widetilde{M}, C^\infty(\widetilde{M}))$ with the additional metric structure $\widetilde{g}$.

If the spacetime is spherically symmetric, we can introduce the relation ρ gluing all points equidistant from the certain point. This is, of course, the equivalence relation. As a result, we obtain the quotient space $(\widetilde{M}/\rho, C^\infty(\widetilde{M})/\rho)$, which is also a differential space [7]. Therefore, the below proposition was proved.

Proposition 2.2. *A Penrose diagram is a differential space.*

We will also prove that any Penrose diagram is also a subcartesian differential space.

Definition 2.3. If (M, C) is a differential space, which has the Hausdorff property and for every point $x \in M$ there exists a neighbourhood $U \ni \tau$ diffeomorphic to some $V \in \mathbb{R}^n$, then (M, C) will be called a subcartesian differential space (see [25]).

A diffeomorphism in the above definition is understood in the following way. Suppose that (M, C) and (N, D) are differential spaces and $F : M \to N$ is a bijecitve (i.e., both injective and surjective) mapping. If (1) for every $f \in D$ it holds $f \circ F \in C$ and (2) for every $g \in C$ it holds $g \circ F^{-1} \in D$, then F will be called a differomorphism from (M, C) to (N, D), or just from M to N.

Proposition 2.4. *A Penrose diagram is a subcartesian differential space.*

Proof. Let $(M, C^\infty(M))$ be a differential space equipped with the additional metric structure given by g, as previously.

Let F be the conformal rescalling. In particular, $F : M \to \widetilde{M} \in \mathbb{R}^n$ for some $n \in \mathbb{N}$. Moreover, the metric g is mapped to $\widetilde{g} := \Omega g$. Then, $(\widetilde{M}, C^\infty(\widetilde{M}))$ is a differential space with the additional metric structure $\widetilde{g}$ and $C^\infty(\widetilde{M}) = F(C^\infty(M))$ and $F(M) = \widetilde{M} \subset \mathbb{R}^n$. Indeed, if $C^\infty(M) = (\mathrm{sc}\{f_1, \ldots, f_n\})_M$, then $C^\infty(\widetilde{M}) = (\mathrm{sc}\{f_1 \circ F^{-1}, \ldots, f_n \circ F^{-1}\})_{\widetilde{M}}$.

Because the spacetime is spherically symmetric, we can, as previously, introduce the relation ρ gluing all points equidistant from the certain point. This is, of course, the equivalence relation. As a result, we obtain the quotient space $(\widetilde{M}/\rho, C^\infty(\widetilde{M})/\rho)$, which is also a differential space [7].

Finally, $\widetilde{M} \subset \mathbb{R}^n$ and $(\mathbb{R}^n, C^\infty(\mathbb{R}^n))$ is a differential space. Moreover, $C^\infty(\widetilde{M}) = (\mathrm{sc}\{\pi_1|_{\widetilde{M}}, \ldots, \pi_n|_{\widetilde{M}}\})_{\widetilde{M}}$, where π_i, $i = 1, \ldots, n$ are canonical projections $\pi_i : \mathbb{R}^n \ni (x_1, \ldots, x_i, \ldots, x_n) \mapsto x_i \in \mathbb{R}$. □

3. Conclusions

Detailed calculations of conformal rescalling functions for Minkowski spacetime and Roberston–Walker universe are, for example, in [3]. Much more thorough description of subcartesian differential spaces is, for example, in [24] and [27]. Some modification of differential spaces is described by Heller and Sasin [13]. A description of cosmological and other applications of differential spaces is present in [8] and [6].

Indeed, differential spaces are conceptually very useful in studying spacetime singularities. For example, the reasoning presented herein can be easily repeated for quasi–regular singularities [10, 15, 17], as they can be described in certain sense as quotient differential spaces, within a similar methodology as the one presented in this paper. Indeed, a spacetime with quasi–regular singularities is the orbit space of the action of a discrete isometry group, i.e., M/G. The set of singular points S is a subset of the space M, whose isometry group is non–trivial. If M is strongly causal, then S is a boundary set (i.e., each point of S is a boundary point of M) if and only if the isometry group (if discrete) is finite. On the other hand, if M is strongly causal and its isometry group is discrete, then S is a boundary set if and only if the isometry group is finite [19]. Indeed, in [19] more details are presented.

References

[1] G. D. Birkhoff and R.E. Langer, Relativity and Modern Physics, Harvard University Press, Cambridge MA, 1923.

[2] G. Börner, The Early Universe, Facts and Fiction, Springer, Berlin, 2003.

[3] S. Carroll, Spacetime and Geometry, Addison Wesley, Boston MA, 2004.

[4] Deepmala and L.N. Mishra, Differential operators over modules and rings as a path to the generalized differential geometry, *Facta Universitatis (Niš) Ser. Math. Inform.* **30** *(2015), 753-764.*

[5] R. D'Inverno, Introducing Einstein's Relativity, Oxford University Press, Oxford, 1992.

[6] K. Drachal, Advantages of generalizing manifold model in mechanics and cosmology, In M. Lieskovský and M. Mokryš, (eds.), Proceedings in Advanced Research in Scientific Areas, The 1st Virtual International Conference, 1451-1453, EDIS – Publishing Institution of the University of Žilina, Žilina, 2012.

[7] K. Drachal, Introduction to d–spaces, *Math. Aeterna* **3***(2013), 753-770.*

[8] K. Drachal, Differential spaces and cosmological models, In S. Badura, M. Mokryš, and A. Lieskovský, (eds.), Proceedings in Scientific Conference, SCIECONF 2014, 376-380, EDIS – Publishing Institution of the University of Žilina, Žilina, 2014.

[9] K. Drachal, Differential spaces and spacetime singularities: a current perspective, *In press.*

[10] G. F. R. Ellis and B. G. Schmidt, Classification of singular space–times, *Gen. Rel. Grav.* **10***(1979), 989-997.*

[11] S. W. Hawking and G. F. R. Ellis, The Large Scale Structure of Space–Time, Cambridge University Press, Cambridge, 1973.

[12] P. G. Fre, Gravity, a Geometrical Course, Springer, Berlin, 2013.

[13] M. Heller and W. Sasin, Structured spaces and their application to relativistic physics, *J. Math. Phys.* **36***(1995), 3644-3662.*

[14] J. T. Jebsen, Über die allgemeinen kugelsymmetrischen Lösungen der Einsteinschen Gravitationsgleichungen im Vakuum, *Ark. Mat. Ast. Fys.* **15***(1921), 1-9.*

[15] S. Krasnikov, Quasiregular singularities taken seriously, *Preprint, arXiv:0909.4963.*

[16] E. Minguzzi and M. Sanchez, The causal hierarchy of spacetimes, In H. Baum and D. Alekseevsky, (eds.), Recent Developments in Pseudo–Riemannian Geometry, 299-358, Eur. Math. Soc. Publ. House, Zürich, 2008.

[17] V. N. Mishra, Some Problems on Approximations of Functions in Banach Spaces, Indian Institute of Technology, Roorkee, 2007.

[18] C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation, W. H. Freeman and Company, New York, 1973.

[19] Z. Odrzygóźdź, Geometrical Properties of Quasi–regular Singularities, Warsaw University of Technology, Warsaw, 1996.

[20] A. R. Parry, A survey of spherically symmetric spacetimes, *Anal. Math. Phys.* **4***(2014), 333-375.*

[21] L. I. Piscoran and V. N. Mishra, Projectively flatness of a new class of (α, β)-metrics, *Georgian Math. Journal, In press.*

[22] R. Sikorski, Abstract covariant derivative, *Colloq. Math.* **18***(1967), 251-272.*

[23] R. Sikorski, Differential modules, *Colloq. Math.* **24***(1972), 45-79.*

[24] J. Śniatycki, Orbits of families of vector fields on subcartesian spaces, *Ann. Inst. Fourier* **53***(2003), 2257-2296.*

[25] J. Śniatycki, Differential Geometry of Singular Spaces and Reduction of Symmetry, Cambridge University Press, Cambridge, 2013.

[26] R. M. Wald, General Relativity, The University of Chicago Press, Chicago IL, 1984.

[27] J. Watts, Differential spaces, vector fields, and orbit–type stratifications, *Preprint, arXiv:1104.4084.*