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Vector Valued multiple of χ^2 over $p-$metric sequence spaces defined by Musielak

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Abstract. In this article, we define the vector valued multiple of χ^2 over $p-$ metric sequence spaces defined by Musielak and study some of their topological properties and some inclusion results.

Keywords: Analytic sequence, double sequences, χ^2 space,Musielak - modulus function, $p-$ metric space, multiplier space.

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1. Introduction

Consider w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication. Throughout this article the space of regularly gai multiple sequence defined over a semi-normed space (X, q), semi-normed by q will be denoted by $\chi^{2R}_{mn}(q)$ and $\Lambda^{2R}_{mn}(q)$ For $X = \mathbb{C}$, the field of complex numbers, these spaces represent the corresponding scalar sequence spaces. Some initial works on double sequence spaces is found in Bromwich [1]. Later on, they were investigated by Hardy [2], Moricz [3], Moricz and Rhoades [4], Basarir and Solankan [5], Tripathy [6], Turkmenoglu [7], and many others. We procure the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{ (x_{mn}) \in w^2 : sup_{m,n \in N} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_p(t) :=$$
$$\left\{ (x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn} - l|^{t_{mn}} = 1 \, for\, some\, l \in \mathbb{C} \right\},$$

$$\mathcal{C}_{0p}(t) := \left\{ (x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1 \right\},$$

$$\mathcal{L}_u(t) := \left\{ (x_{mn}) \in w^2 : \textstyle\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [8,9] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [10] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [11] and Tripathy have independently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Altay and Basar [12] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Basar and Sever [13] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [14] have studied the space $\chi^2_M(p,q,u)$ of double sequences and gave some inclusion relations.

The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [15] as an extension of the definition of strongly Cesàro summable sequences. Cannor [16] further extended this definition to a definition of strong $A-$ summability with respect to a modulus where $A=(a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong $A-$ summability with respect to a modulus, and $A-$ statistical convergence. In [17] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [18]-[19], and [20] the four dimensional matrix transformation $(Ax)_{k,\ell}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{k\ell}^{mn}x_{mn}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a,b,\geq 0$ and $0<p<1$, we have

$$(a+b)^p \leq a^p+b^p \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty}x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn}=\sum_{i,j=1}^{m,n}x_{ij}(m,n\in\mathbb{N})$.

A sequence $x=(x_{mn})$is said to be double analytic if, $sup_{mn}\left|x_{mn}\right|^{1/m+n}<\infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x=(x_{mn})$ is called double gai sequence if $\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n}\to 0$ as $m,n\to\infty$. The double gai sequences will be denoted by χ^2. Let $\phi=\{finite\ sequences\}$. Consider a double sequence $x=(x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]}=\sum_{i,j=0}^{m,n}x_{ij}\Im_{ij}$ for all $m,n\in\mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i,j\in\mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]}\to x$.

t An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x=(x_k)\to(x_{mn})(m,n\in\mathbb{N})$ are also continuous.

Let M and Φ are mutually complementary modulus functions. Then, we have:

(i) For all $u,y\geq 0$,

$$uy\leq M(u)+\Phi(y),(Young's\,inequality)[See[21]] \tag{1.2}$$

(ii) For all $u \geq 0$,
$$u\eta(u) = M(u) + \Phi(\eta(u)). \quad (1.3)$$

(iii) For all $u \geq 0$, and $0 < \lambda < 1$,
$$M(\lambda u) \leq \lambda M(u) \quad (1.4)$$

Lindenstrauss and Tzafriri [22] used the idea of Orlicz function to construct Orlicz sequence space
$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \, for\, some\, \rho > 0 \right\},$$
The space ℓ_M with the norm
$$\|x\| = inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$
becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p\,(1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.
A sequence $f = (f_{mn})$ of modulus function is called a Musielak-modulus function. A sequence $g = (g_{mn})$ defined by
$$g_{mn}(v) = sup\left\{ |v|\, u - (f_{mn})(u) : u \geq 0 \right\}, m, n = 1, 2, \cdots$$
is called the complementary function of a Musielak-modulus function f. For a given Musielak modulus function f, the Musielak-modulus sequence space t_f is defined as follows
$$t_f = \left\{ x \in w^2 : M_f\left(|x_{mn}|\right)^{1/m+n} \to 0\, as\, m, n \to \infty \right\},$$
where M_f is a convex modular defined by
$$M_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn}\left(|x_{mn}|\right)^{1/m+n}, x = (x_{mn}) \in t_f.$$
We consider t_f equipped with the Luxemburg metric
$$d(x, y) = sup_{mn}\left\{ inf\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn}\left(\frac{|x_{mn}|^{1/m+n}}{mn}\right)\right) \leq 1 \right\}$$
If X is a sequence space, we give the following definitions:

(i)X'= the continuous dual of X;

(ii)$X^{\alpha} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn} x_{mn}| < \infty, \, for\, each\, x \in X \right\};$

(iii)$X^{\beta} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn} x_{mn}\, is\, convegent, \, foreach\, x \in X \right\};$

(iv)$X^{\gamma} = \left\{ a = (a_{mn}) : sup_{mn \geq 1} \left| \sum_{m,n=1}^{M,N} a_{mn} x_{mn} \right| < \infty, foreachx \in X \right\};$

(v)$let\, X\, be\, an\, FK-space \supset \phi; \, then\, X^f = \left\{ f(\Im_{mn}) : f \in X' \right\};$

(vi)$X^{\delta}=\left\{a=(a_{mn}): sup_{mn}\left|a_{mn}x_{mn}\right|^{1/m+n}<\infty,\, foreach\, x\in X\right\};$

$X^{\alpha}, X^{\beta}, X^{\gamma}$ are called $\alpha-(orK\ddot{o}the-Toeplitz)$dual of $X, \beta-(or\, generalized-K\ddot{o}the-Toeplitz)\, dual\, of\, X, \gamma-dual\, of\, X,\, \delta-dual\, of\, X\, respectively. X^{\alpha}$ is defined by Gupta and Kampthan . It is clear that $X^{\alpha}\subset X^{\beta}$ and $X^{\alpha}\subset X^{\gamma}$, but $X^{\beta}\subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz as follows

$$Z(\Delta)=\{x=(x_k)\in w:(\Delta x_k)\in Z\}$$

for $Z=c, c_0$ and ℓ_{∞}, where $\Delta x_k=x_k-x_{k+1}$ for all $k\in\mathbb{N}$.

Here c, c_0 and ℓ_{∞} denote the classes of convergent,null and bounded scalar valued single sequences respectively. The difference sequence space bv_p of the classical space ℓ_p is introduced and studied in the case $1\leq p\leq\infty$ by Başar and Altay and in the case $0<p<1$ by Altay and Başar. The spaces $c(\Delta), c_0(\Delta), \ell_{\infty}(\Delta)$ and bv_p are Banach spaces normed by

$\|x\|=|x_1|+sup_{k\geq 1}|\Delta x_k|$ and $\|x\|_{bv_p}=\left(\sum_{k=1}^{\infty}|x_k|^p\right)^{1/p}, (1\leq p<\infty).$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta)=\left\{x=(x_{mn})\in w^2:(\Delta x_{mn})\in Z\right\}$$

where $Z=\Lambda^2, \chi^2$ and $\Delta x_{mn}=(x_{mn}-x_{mn+1})-(x_{m+1n}-x_{m+1n+1})=x_{mn}-x_{mn+1}-x_{m+1n}+x_{m+1n+1}$ for all $m, n\in\mathbb{N}$.

2. Definition and Preliminaries

Throughout this article a multiple sequence is denoted by;
$A=\langle a_{m_1m_2\cdots m_rn_1n_2\cdots n_s}\rangle$, a multiple infinite array of elements
$a_{m_1m_2\cdots m_rn_1n_2\cdots n_s}\in X$ for all $m_1m_2\cdots m_rn_1n_2\cdots n_s\in\mathbb{N}$.
Let $n\in\mathbb{N}$ and X be a real vector space of dimension m, where $n\leq m$.
A real valued function $d_p(x_1,\ldots,x_n)=\|(d_1(x_1),\ldots,d_n(x_n))\|_p$ on X satisfying the following four conditions:

(i) $\|(d_1(x_1),\ldots,d_n(x_n))\|_p=0$ if and and only if $d_1(x_1),\ldots,d_n(x_n)$ are linearly dependent,

(ii) $\|(d_1(x_1),\ldots,d_n(x_n))\|_p$ is invariant under permutation,

(iii) $\|(\alpha_{m_1m_2\cdots m_rn_1n_2\cdots n_s}d_1(x_1),\ldots,\alpha_{m_1m_2\cdots m_rn_1n_2\cdots n_s}d_n(x_n))\|_p=$
$|\alpha_{m_1m_2\cdots m_rn_1n_2\cdots n_s}|\,\|(d_1(x_1),\ldots,d_n(x_n))\|_p$,
$\alpha_{m_1m_2\cdots m_rn_1n_2\cdots n_s}\in\mathbb{R}$

(iv) $d_p((x_1,y_1),(x_2,y_2)\cdots(x_n,y_n))=(d_X(x_1,x_2,\cdots x_n)^p$
$+\,d_Y(y_1,y_2,\cdots y_n)^p)^{1/p} for 1\leq p<\infty$; (or)

(v) $d((x_1,y_1),(x_2,y_2),\cdots(x_n,y_n)):=\sup\{d_X(x_1,x_2,\cdots x_n), d_Y(y_1,y_2,\cdots y_n)\}$,
for $x_1,x_2,\cdots x_n\in X, y_1,y_2,\cdots y_n\in Y$ is called the p product metric of

the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces.
A trivial example of p product metric of n metric space is the p norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\|(d_1(x_1), \ldots, d_n(x_n))\|_E = sup\left(|det(d_{mn}(x_{mn}))|\right) = sup\left(\begin{vmatrix} d_{11}(x_{11}) & d_{12}(x_{12}) & \ldots & d_{1n}(x_{1n}) \\ d_{21}(x_{21}) & d_{22}(x_{22}) & \ldots & d_{2n}(x_{1n}) \\ . & & & \\ . & & & \\ . & & & \\ d_{n1}(x_{n1}) & d_{n2}(x_{n2}) & \ldots & d_{nn}(x_{nn}) \end{vmatrix}\right)$$

where $x_i = (x_{i1}, \cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots n$.
If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the $p-$ metric. Any complete $p-$ metric space is said to be $p-$ Banach metric space.

2.1. **Definition.** Let X be a linear metric space. A function $\rho : X \to \mathbb{R}$ is called paranorm, if
(1) $\rho(x) \geq 0$, for all $x \in X$;
(2) $\rho(-x) = \rho(x)$, for all $x \in X$;
(3) $\rho(x + y) \leq \rho(x) + \rho(y)$, for all $x, y \in X$;
(4) If (σ_{mn}) is a sequence of scalars with $\sigma_{mn} \to \sigma$ as $m, n \to \infty$ and (x_{mn}) is a sequence of vectors with $\rho(x_{mn} - x) \to 0$ as $m, n \to \infty$, then $\rho(\sigma_{mn}x_{mn} - \sigma x) \to 0$ as $m, n \to \infty$.
A paranorm w for which $\rho(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, w) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [23], Theorem 10.4.2, p.183).

2.2. **Definition.** A multiple sequence space E is said to be solid if $\langle \alpha_{m_1m_2\cdots m_rn_1n_2\cdots n_s} a_{m_1m_2\cdots m_rn_1n_2\cdots n_s} \rangle \in E$ whenever $\langle a_{m_1m_2\cdots m_rn_1n_2\cdots n_s} \rangle \in E$ for all multiple sequences $\langle \alpha_{m_1m_2\cdots m_rn_1n_2\cdots n_s} \rangle$ of scalars with $|\alpha_{m_1m_2\cdots m_rn_1n_2\cdots n_s}| \leq 1$ for all $m_1m_2 \cdots m_rn_1n_2 \cdots n_s \in \mathbb{N}$.

2.3. **Definition.** A multiple sequence space E is said to be symmetric if $\langle a_{m_1m_2\cdots m_rn_1n_2\cdots n_s} a_{m_1m_2\cdots m_rn_1n_2\cdots n_s} \rangle \in E$, implies $\left\langle a_{\pi(m_1m_2\cdots m_rn_1n_2\cdots n_s)} \right\rangle \in E$,
where $\pi(m_1m_2 \cdots m_rn_1n_2 \cdots n_s) \in E$ are permutations of $(M \times N) \cdots (M \times N)$.

2.4. **Definition.** A multiple sequence space E is said to be monotone if it contains the canonical pre-images of all its step spaces.

2.5. **Defintion.** A multiple sequence space E is said to be convergence free if $\langle b_{m_1m_2\cdots m_rn_1n_2\cdots n_s}\rangle \in E$, whenever $\langle a_{m_1m_2\cdots m_rn_1n_2\cdots n_s}\rangle \in E$ and $b_{m_1m_2\cdots m_rn_1n_2\cdots n_s} = 0$ whenver $a_{m_1m_2\cdots m_rn_1n_2\cdots n_s} = 0$.

2.6. **Remark.** A sequence space E is solid implies E is monotone.

Let f be an Musielak modulus function. Now we introduce the following multiple sequence spaces:

$$\left[\Lambda^{2fq}_{mn}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right] = \langle a_{m_1m_2\cdots m_rn_1n_2\cdots n_s}\rangle \in w^{2q}_{mn} :$$
$$\sup_{m_1m_2\cdots m_rn_1n_2\cdots n_s}$$
$$\left[f\left(q\left(\eta_{uv}(x), \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right)\right] < \infty,$$
where $\eta_{uv}(x) = |a_{m_1m_2\cdots m_rn_1n_2\cdots n_s}|^{(1/m_1m_2\cdots m_r)+n_1n_2\cdots n_s}$

$$\left[\chi^{2fq}_{mn}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right] = \langle a_{m_1m_2\cdots m_rn_1n_2\cdots n_s}\rangle \in w^{2q}_{mn} :$$
$$\left[f\left(q\left(\mu_{uv}(x), \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right)\right] \to 0;$$
$as\, m_1m_2\cdots m_rn_1n_2\cdots n_s \to \infty$.
where $\mu_{uv}(x)\,((m_1m_2\cdots m_r + n_1n_2\cdots n_s)!\,|a_{m_1m_2\cdots m_rn_1n_2\cdots n_s}|)^{(1/m_1m_2\cdots m_r)+n_1n_2\cdots n_s}$
$A = \langle a_{m_1m_2\cdots m_rn_1n_2\cdots n_s}\rangle \in \chi^{2Rf}_{mn}(q)$, (i.e)., regularly gai if $\langle a_{m_1m_2\cdots m_rn_1n_2\cdots n_s}\rangle \in \chi^{2f}_{mn}(q)$ and the following limit hold:

$$\left[f\left(q\left(\mu_{uv}(x), \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right)\right] \to 0; as\, m_1n_1 \to \infty$$
and $m_2\cdots m_r, n_2\cdots n_s \in \mathbb{N}$
$$\left[f\left(q\left(\mu_{uv}(x), \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right)\right] \to 0;$$
$as\, m_2n_2 \to \infty$ and $m_3\cdots m_r, n_3\cdots n_s \in \mathbb{N}$

$\vdots$

$\vdots$

$$\vdots\left[f\left(q\left(\mu_{uv}(x), \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right)\right] \to 0\, as\, m_rn_s \to \infty$$
and $m_1\cdots m_{r-1}, n_1\cdots n_{s-1} \in \mathbb{N}$

2.7. **Remark.** The space $\chi^{2Rf}_{mn}(q)$ has the following definition too
$$\left[\chi^{2fq}_{mn}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right] = \langle a_{m_1m_2\cdots m_rn_1n_2\cdots n_s}\rangle \in w^{2q}_{mn} :$$
$$\left[f\left(q\left(\mu_{uv}(x), \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right)\right] \to 0;$$
$as\, max\,\{m_1m_2\cdots m_rn_1n_2\cdots n_s\} \to \infty$. $\chi^{2Bf}_{mn}(q) = \chi^{2f}_{mn}(q) \bigcap \Lambda^{2f}_{mn}(q)$.

3. Main result

Theorem 3.1. *$\chi^{2Rf}_{mn}(q), \chi^{2Bf}_{mn}(q)$ and $\Lambda^{2f}_{mn}(q)$ of multiple sequences are linear spaces.*

Proof: We have to use linearity condition and then prove to the statement. Hence it is trivial. Therefore omit the proof.

Theorem 3.2. *Let $f = (f_{mn})$ be a multiple sequence of Musielak-modulus functions. Then then space* $\left[\chi^{2Rf}_{mn}(q), \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$ *is a paranormed space with respect to the paranorm defined by*
$g(x) = inf$
$\left\{\left(\left[f\left(q\left(\|\mu_{uv}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]\right)\right)\right\} \leq 1.$

Proof: Clearly $g(x) \geq 0$ for $x = (a_{m_1 m_2 \cdots m_r, n_1 n_2 \cdots n_s}) \in \left[\chi^{2Rf}_{mn}(q), \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$. Since $f(0) = 0$, we get $g(0) = 0$.

Conversely, suppose that $g(x) = 0$, then
$inf\left\{\left(\left[f\left(q\left(\|\mu_{uv}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]\right)\right)\right\} \leq 1 = 0.$
Suppose that $\mu_{uv}(x) \neq 0$ for each $u, v \in \mathbb{N}$. Then
$\|\mu_{uv}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \to \infty$. It follows that
$\left(\left[f\left(q\left(\|\mu_{uv}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right)\right]\right) \to \infty$ which is a contradiction. Therefore $\mu_{uv}(x) = 0$. Let
$\left(\left[f\left(q\left(\|\mu_{uv}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right)\right]\right) \leq 1$ and
$\left(\left[f\left(q\left(\|\mu_{uv}(y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right)\right]\right) \leq 1.$

Then by using Minkowski's inequality, we have

$$\left(\left[f\left(q\left(\|\mu_{uv}(x+y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right)\right]\right) \leq$$
$$\left(\left[f\left(q\left(\|\mu_{uv}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right)\right]\right) +$$
$$\left(\left[f\left(q\left(\|\mu_{uv}(y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right)\right]\right).$$

So we have $g(x+y) = inf$
$$\left\{\left(\left[f\left(q\left(\|\mu_{uv}(x+y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right)\right]\right) \leq 1\right\} \leq$$
$$inf\left\{\left(\left[f\left(q\left(\|\mu_{uv}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right)\right]\right) \leq 1\right\} +$$
$$inf\left\{\left(\left[f\left(q\left(\|\mu_{uv}(y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right)\right]\right) \leq 1\right\}$$
Therefore,

$$g(x+y) \leq g(x) + g(y).$$

Finally, to prove that the scalar multiplication is continuous. Let $\lambda_{m_1 m_2 \cdots m_r n_1 n_2 \cdots n_s}$ be any complex number. By definition,
$g(\lambda_{m_1 m_2 \cdots m_r n_1 n_2 \cdots n_s} x) = inf$

$\left\{\left(\left[f\left(q\left(\|\mu_{uv}\left(\lambda_{m_1m_2\cdots m_rn_1n_2\cdots n_s}x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\|_p\right)\right)\right]\right)\leq 1\right\}$
Then
$g\left(\lambda_{m_1m_2\cdots m_rn_1n_2\cdots n_s}x\right)=inf$
$\left\{\left(\left(|\lambda_{m_1m_2\cdots m_rn_1n_2\cdots n_s}|\,t\right):\left(\left[f\left(q\left(\|\mu_{uv}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\|_p\right)\right)\right]\right)\leq 1\right\}\right.$
where $t=\frac{1}{|\lambda|}$. Since $|\lambda|\leq max\left(1,|\lambda|^{suppuv}\right)$, we have
$g\left(\lambda_{m_1m_2\cdots m_rn_1n_2\cdots n_s}x\right)\leq max\left(1,|\lambda|^{suppuv}\right)$
$inf\left\{t:\left(\left[f\left(q\left(\|\mu_{uv}\left(\lambda_{m_1m_2\cdots m_rn_1n_2\cdots n_s}x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\|_p\right)\right)\right]\right)\leq 1\right\}.$
This completes the proof.

Theorem 3.3. *The space* $\left[\chi^{2Rf}_{mn}\left(q\right),\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\|_p\right]$ *is not symmetric.*

Proof: Let $\langle a_{m_1m_2\cdots m_r,n_1n_2\cdots n_s}\rangle\in\left[\chi^{2Rf}_{mn}\left(q\right),\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\|_p\right]$.
Then for a given $\epsilon>0$ there exists a positive integers $g_1,g_2,\cdots g_{g+1}h_1,h_2\cdots h_{h+1}$ such that

$\left[f\left(q\left(\mu_{uv}\left(x\right),\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\|_p\right)\right)\right]<\epsilon$ for all
$m_1n_1>g_1h_1$ for all $m_2\cdots m_r,n_2\cdots n_s\in\mathbb{N}$

$\left[f\left(q\left(\mu_{uv}\left(x\right),\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\|_p\right)\right)\right]<\epsilon$ for all
$m_2n_2>g_2h_2$ for all $m_3\cdots m_r,n_3\cdots n_s\in\mathbb{N}$

$\vdots$

$\vdots$

$\left[f\left(q\left(\mu_{uv}\left(x\right),\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\|_p\right)\right)\right]<\epsilon$
for all $m_rn_s>g_rh_s$ for all $m_1\cdots m_{r-1},n_1\cdots n_{s-1}\in\mathbb{N}$

$\left[f\left(q\left(\mu_{uv}\left(x\right),\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\|_p\right)\right)\right]<\epsilon$ for all
$m_1>g_{r+1},m_2>g_{r+1},\cdots,m_r>g_{r+1},n_1>h_{s+1},n_2>h_{s+1},\cdots n_s>h_{s+1}\in\mathbb{N}$.
Let $g_0h_0=max\left\{g_1,g_2,\cdots g_r,g_{r+1},h_1,h_2\cdots h_r,h_{r+1}\right\}$.
Let $\langle b_{m_1m_2\cdots m_r,n_1n_2\cdots n_s}\rangle$ be a rearrangement of $\langle a_{m_1m_2\cdots m_r,n_1n_2\cdots n_s}\rangle$. Then we have $a_{i_1i_2\cdots i_r,j_1j_2\cdots j_s}=b_{m_{i_1}m_{i_2}\cdots m_{i_r},n_{j_1}n_{j_2}\cdots n_{j_s}}$ for all $i_1i_2\cdots i_r,j_1j_2\cdots j_s\in\mathbb{N}$. Let
$g_{r+2}h_{r+2}=max$
$m_{1_1}m_{2_1}\cdots m_{r_1},m_{r+1_1},m_{(r_0)_1}n_{1_1}n_{2_1}\cdots n_{s_1},n_{s+1_1}\cdots\cdots\cdots m_{1_r}m_{2_r}\cdots m_{r_r},m_{r+1_r},m_{(r_0)_r}$
$n_{1_s}n_{2_s}\cdots n_{s_s},n_{s+1_s}$.
Then we have

$\left[f\left(q\left(\mu_{uv}(x),\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right)\right] < \epsilon$ for all $m_1n_1 > g_{r+2}h_{s+2}\cdots m_rn_s > g_{r+2}h_{s+2}$.

Thus $\left\langle b_{m_{i_1}m_{i_2}\cdots m_{i_r},n_{j_1}n_{j_2}\cdots n_{j_s}}\right\rangle \in \left[\chi^{2Rf}_{mn}(q),\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right]$.

Hence

$\left[\chi^{2Rf}_{mn}(q),\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right]$ is a symmetric space.

Theorem 3.4. *The spaces* $\left[\chi^{2Rf}_{mn}(q),\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right]$ *and* $\left[\chi^{2Bf}_{mn}(q),\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right]$ *are solid*

Proof: The spaces $\left[\chi^{2Rf}_{mn}(q),\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right]$ and $\left[\chi^{2Bf}_{mn}(q),\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right]$ are solid follows the following inequality.

$\left[f\left(q\left(\alpha_{m_1m_2\cdots m_r,n_1n_2\cdots n_s}\mu_{uv}(x),\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right)\right] \leq$

$\left[f\left(q\left(\mu_{uv}(x),\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right)\right)\right]$ for all $m_1m_2\cdots m_r, n_1n_2\cdots n_s \in \mathbb{N}$ and scalars $\langle\alpha_{m_1m_2\cdots m_r,n_1n_2\cdots n_s}\rangle$ with $|\alpha_{m_1m_2\cdots m_r,n_1n_2\cdots n_s}| \leq 1$ for all $m_1m_2\cdots m_r, n_1n_2\cdots n_s \in \mathbb{N}$.

Theorem 3.5. *The spaces* $\left[\chi^{2Rf}_{mn}(q),\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right]$ *and* $\left[\chi^{2Bf}_{mn}(q),\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right]$ *are monotone.*

Proof: The proof follows from the Remark 2.6 and Theorem 3.4.

Theorem 3.6. *Let f_1 and f_2 be multiple sequence of Musielak modulus functions. Then we have*

(1) $\left[\chi^{2Rf_1}_{mn},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right] \subseteq \left[\chi^{2Rf_2\circ f_1}_{mn},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right]$

(2) $\left[\chi^{2Rf_1}_{mn},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right] \bigcap \left[\chi^{2Rf_2}_{mn},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right] \subseteq$

$\left[\chi^{2Rf_1+f_2}_{mn},\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right]$

(3) $\left[\chi^{2Rf_1}_{mn}(q_1),\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right] \bigcap \left[\chi^{2Rf_1}_{mn}(q_2),\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right] \subseteq$

$\left[\chi^{2Rf_1}_{mn}(q_1+q_2),\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right]$ *where q_1 and q_2 are two semi norms.*

(4) If q_1 is stronger than q_2, then

$\left[\chi^{2Rf_1}_{mn}(q_1),\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right] \subseteq \left[\chi^{2Rf_1}_{mn}(q_2),\|(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p\right]$.

Proof: We have to take one condition and then easily prove to other condition. Hence it is trivial. Therefore omit the proof.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this research paper.

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References

[1] T.J.I'A. Bromwich, An introduction to the theory of infinite series, Macmillan and Co.Ltd., New York, (1965).

[2] G.H. Hardy, On the convergence of certain multiple series, Proc. Camb. Phil. Soc., 19 (1917), 86-95.

[3] F. Moricz, Extentions of the spaces c and c_0 from single to double sequences, Acta. Math. Hung., 57(1-2), (1991), 129-136.

[4] F. Moricz and B.E. Rhoades, Almost convergence of double sequences and strong regularity of summability matrices, Math. Proc. Camb. Phil. Soc., 104, (1988), 283-294.

[5] M. Basarir and O. Solancan, On some double sequence spaces, J. Indian Acad. Math., 21(2) (1999), 193-200.

[6] B.C. Tripathy, On statistically convergent double sequences, Tamkang J. Math., 34(3), (2003), 231-237.

[7] A. Turkmenoglu, Matrix transformation between some classes of double sequences, J. Inst. Math. Comp. Sci. Math. Ser., 12(1), (1999), 23-31.

[8] A. Gökhan and R. Çolak, The double sequence spaces $c_2^P(p)$ and $c_2^{PB}(p)$, Appl. Math. Comput., 157(2), (2004), 491-501.

[9] A. Gökhan and R. Çolak, Double sequence spaces ℓ_2^∞, ibid., 160(1), (2005), 147-153.

[10] M. Zeltser, Investigation of Double Sequence Spaces by Soft and Hard Analitical Methods, Dissertationes Mathematicae Universitatis Tartuensis 25, Tartu University Press, Univ. of Tartu, Faculty of Mathematics and Computer Science, Tartu, 2001.

[11] M. Mursaleen and O.H.H. Edely, Statistical convergence of double sequences, J. Math. Anal. Appl., 288(1), (2003), 223-231.

[12] B. Altay and F. Baąar, Some new spaces of double sequences, J. Math. Anal. Appl., 309(1), (2005), 70-90.

[13] F. Başar and Y. Sever, The space $\mathcal{L}_p$ of double sequences, Math. J. Okayama Univ, 51, (2009), 149-157.

[14] N. Subramanian and U.K. Misra, The semi normed space defined by a double gai sequence of modulus function, Fasciculi Math., 46, (2010).

[15] I.J. Maddox, Sequence spaces defined by a modulus, Math. Proc. Cambridge Philos. Soc, 100(1) (1986), 161-166.

[16] J. Cannor, On strong matrix summability with respect to a modulus and statistical convergence, Canad. Math. Bull., 32(2), (1989), 194-198.
[17] A. Pringsheim, Zurtheorie derzweifach unendlichen zahlenfolgen, Math. Ann., 53, (1900), 289-321.
[18] H.J. Hamilton, Transformations of multiple sequences, Duke Math. J., 2, (1936), 29-60.
[19] H.J. Hamilton, A Generalization of multiple sequences transformation, Duke Math. J., 4, (1938), 343-358.
[20] H.J. Hamilton, Preservation of partial Limits in Multiple sequence transformations, Duke Math. J., 4, (1939), 293-297.
[21] P.K. Kamthan and M. Gupta, Sequence spaces and series, Lecture notes, Pure and Applied Mathematics, 65 Marcel Dekker, In c., New York , 1981.
[22] J. Lindenstrauss and L. Tzafriri, On Orlicz sequence spaces, Israel J. Math., 10 (1971), 379-390.
[23] A. Wilansky, Summability through Functional Analysis, North-Holland Mathematical Studies, North-Holland Publishing, Amsterdam, Vol.85(1984).