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Existence and uniqueness of solutions for neutral periodic integro-differential equations with infinite delay

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Abstract. We prove the existence of solutions for the neutral periodic integro-differential equation with infinite delay

$$x'(t) = G(t, x(t), x(t-\tau(t))) + \frac{d}{dt}Q(t, x(t-\tau(t))) + \int_{-\infty}^{t} \Big(\sum_{j=1}^{n} g_j(t,s)\Big) f(x(s))ds,$$

$$x(t+T) = x(t).$$

A Krasnoselskii and Banach's fixed point theorems are employed in establishing our results.

Keywords: Krasnoselskii's Fixed point theorem, integro-differential neutral equation, periodic solution.

2000 Mathematics subject classification: 34A37, 34A12, 39A05.

1. Introduction

In this paper, we consider the neutral integro-differential equation

$$x'(t) = G(t, x(t), x(t-\tau(t))) + \frac{d}{dt}Q(t, x(t-\tau(t))) + \int_{-\infty}^{t} \Big(\sum_{j=1}^{n} g_j(t,s)\Big) f(x(s))ds,$$

$$x(t+T) = x(t), \tag{1.1}$$

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where $Q : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $G : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ and $g_j : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ for $j = 1, ..., n$ are continuous in their respective arguments.

This work is mainly motivated by the work of Althubiti, Makhzoum and Raffoul [1], in which they obtained sufficient conditions for the existence of periodic solutions for the equation

$$x'(t) = -a(t)x(t) + \frac{d}{dt}Q(t, x(t-\tau(t))) + \int_{-\infty}^{t} D(t,s)f(x(s))ds. \tag{1.2}$$

We refer to [1]-[17], and [19] for some qualitative results on neutral differential equations, integral equations and integro-differential equations.

The rest of the paper is organized as follows. In section 2, we provide some preliminary material needed for our work and in section 3 we state and prove our main results.

2. Preliminaries

Let $T > 0$ and define the set $P_T = \{\phi \in C(\mathbb{R}, \mathbb{R}) : \phi(t+T) = \phi(t)\}$, where C is the space of continuous real valued functions. Then $(P_T, ||.||)$ is a Banach space when it is endowed with the supremum norm $\|x\| = \sup_{t\in[0,T]} |x(t)|$.

In this paper we make the following assumptions.

$$\begin{aligned} &g_j(t+T, s+T) = g_j(t,s), \text{ for } j = 1, 2, ..., n, \\ &Q(t+T, x) = Q(t,x), \quad G(t+T, x, y) = G(t,x,y), \\ &\tau(t+T) = \tau(t). \end{aligned} \tag{2.1}$$

Also, there exist a continuous function $h : \mathbb{R} \to \mathbb{R}$ such that

$$h(t+T) = h(t), \quad \int_0^T h(s)ds > 0. \tag{2.2}$$

We further assume that the functions $Q(t,x)$, $G(t,x,y)$ and $f(x)$ are globally Lipschitz. That is, there exist positive constants K_1, K_2, K_3, K_4 such that

$$| Q(t,x) - Q(t,y) | \leq K_1 \parallel x - y \parallel, \tag{2.3}$$

$$| G(t,x,y) - G(t,w,z) | \leq K_2 \parallel x - w \parallel + K_3 \parallel y - z \parallel . \tag{2.4}$$

and

$$| f(x) - f(y) | \leq K_4 \parallel x - y \parallel . \tag{2.5}$$

Also, there exist a constant K_5 such that

$$\int_{-\infty}^{t} \Big| \sum_{j=1}^{n} g_j(t,u) \Big| du < K_5 < \infty. \tag{2.6}$$

Lemma 2.1. *Suppose* (2.1) *hold. Let* $h : \mathbb{R} \to \mathbb{R}$ *be an arbitrary continuous function such that* (2.2) *also hold. If* $x(t) \in P_T$, *then* $x(t)$ *is a solution of equation* (1.1) *if and only if*

$$\begin{aligned} x(t) &= Q(t, x(t-\tau(t)) + \big(1 - e^{-\int_{t-T}^{t} h(u)du}\big)^{-1} \\ &\quad \times \int_{t-T}^{t} \Big[- h(s)Q(s, x(s-\tau(s))) \\ &\quad + h(s)x(s) + \int_{-\infty}^{s} \Big(\sum_{j=1}^{n} g_j(s,u) \Big) f(x(u))du \\ &\quad + G(s, x(s), x(s-\tau(s))) \Big] e^{-\int_s^t h(u)du} ds\,. \end{aligned} \tag{2.7}$$

Proof. Let $x(t) \in P_T$ be a solution of (1.1). Rewrite (1.1) as

$$\begin{aligned} \Big(x(t) - Q(t, x(t-\tau(t)))\Big)' &= -h(t)[x(t) - Q(t, x(t-\tau(t)))] - h(t)Q(t, x(t-\tau(t))) \\ &\quad + h(t)x(t) + \int_{-\infty}^{t} \Big(\sum_{j=1}^{n} g_j(t,s) \Big) f(x(s))ds \\ &\quad + G(t, x(t), x(t-\tau(t))). \end{aligned} \tag{2.8}$$

Multiply both sides of (2.8) by $e^{\int_0^t h(u)du}$ and then integrate from $t-T$ to t to obtain

$$\begin{aligned} &\int_{t-T}^{t} \Big[\Big(x(s) - Q(s, x(s-\tau(s))) \Big) e^{\int_0^s h(u)du} \Big]' ds \\ &= \int_{t-T}^{t} \Big[- h(s)Q(s, x(s-\tau(s))) \\ &\quad + h(s)x(s) + \int_{-\infty}^{s} \Big(\sum_{j=1}^{n} g_j(s,u) \Big) f(x(u))du + G(s, x(s), x(s-\tau(s))) \Big] e^{\int_0^s h(u)du} ds. \end{aligned}$$

Thus we obtain,

$$\begin{aligned}&\Big[(x(t)-Q(t,x(t-\tau(t)))e^{\int_0^t h(u)du}\\&\quad-(x(t-T)-Q(t-T,x(t-T-\tau(t-T)))e^{\int_0^{t-T} h(u)du}\\&=\int_{t-T}^{t}\Big[-h(s)Q(s,x(s-\tau(s)))\\&\quad+h(s)x(s)+\int_{-\infty}^{s}\Big(\sum_{j=1}^{n}g_j(s,u)\Big)f(x(u))du\\&\quad+G(s,x(s),x(s-\tau(s)))\Big]e^{\int_0^s h(u)du}ds.\end{aligned}$$

By dividing both sides of the above equation by $\exp(\int_0^t h(u)du)$ and using the fact that $x(t)=x(t-T)$ together with condition (2.1), we obtain the desired result.
Since each step in the above work is reversible, the proof is complete.

□

We next state Krasnoselskii's Theorem which can be found in [18].

Theorem 2.2. *(Krasnoselskii's) Let* $\mathbb{M}$ *be a closed convex nonempty subset of a Banach space* $(\mathbb{S},||.||)$. *Suppose that* J *and* H *map* $\mathbb{M}$ *into* $\mathbb{S}$ *such that*
(i) $x,y\in\mathbb{M}$, *implies* $Jx+Hy\in\mathbb{M}$,
(ii) H *is continuous and* $H\mathbb{M}$ *is contained in a compact set,*
(iii) J *is a contraction mapping.*
Then there exists $z\in\mathbb{M}$ *with* $z=Jz+Hz$.

Define the mappings $J:P_T\to P_T$ and $H:P_T\to P_T$ by

$$(Jx)(t) = Q(t,x(t-\tau(t))), \tag{2.9}$$

and

$$\begin{aligned}(Hx)(t) &= \big(1-e^{-\int_{t-T}^{t}h(u)du}\big)^{-1}\int_{t-T}^{t}\Big[-h(s)Q(s,x(s-\tau(s)))\\&\quad+h(s)x(s)+\int_{-\infty}^{s}\Big(\sum_{j=1}^{n}g_j(s,u)\Big)f(x(u))du\\&\quad+G(s,x(s),x(s-\tau(s)))\Big]e^{-\int_s^t h(u)du}ds\end{aligned} \tag{2.10}$$

respectively.

3. Main result

In this section we state and prove our main results.

Lemma 3.1. *Assume that* (2.1), (2.3)-(2.6) *hold. Assume further that there exist a continuous function* $h : \mathbb{R} \to \mathbb{R}$ *such that* (2.2) *is satisfied. Then* $H : P_T \to P_T$, *as defined by* (2.10), *is continuous and compact.*

Proof. We will first show that $H : P_T \to P_T$, where $(P_T, ||.||)$ is a Banach space. It must be noted that a subset of P_T which is closed and convex is defined in Theorem 3.3 and is denoted by $\mathbb{M}$. Evaluating (2.10) at $T + t$ we obtain,

$$\begin{aligned}(Hx)(t+T) &= \big(1 - e^{-\int_t^{t+T} h(u)du}\big)^{-1} \int_t^{t+T} \Big[- h(s)Q(s, x(s-\tau(s))) \\ &\quad + h(s)x(s) + \int_{-\infty}^{s} \Big(\sum_{j=1}^{n} g_j(s,u)\Big) f(x(u))du \\ &\quad + G(s, x(s), x(s-\tau(s)))\Big] e^{-\int_s^{t+T} h(u)du} ds.\end{aligned}$$

With $k = s - T$ and $v = u - T$ we obtain,

$$\begin{aligned}(Hx)(t+T) &= \big(1 - e^{-\int_t^{t+T} h(u)du}\big)^{-1} \int_t^{t+T} \Big[- h(s)Q(s, x(s-\tau(s))) \\ &\quad + h(s)x(s) + \int_{-\infty}^{s} \Big(\sum_{j=1}^{n} g_j(s,u)\Big) f(x(u))du \\ &\quad + G(s, x(s), x(s-\tau(s)))\Big] e^{-\int_s^{t+T} h(u)du} ds \\ &= \big(1 - e^{-\int_{t-T}^{t} h(v+T)dv}\big)^{-1} \int_{t-T}^{t} \Big[- h(k+T)Q(k+T, x(k+T-\tau(k+T))) \\ &\quad + h(k+T)x(k+T) + \int_{-\infty}^{k+T} \Big(\sum_{j=1}^{n} g_j(k+T, v+T)\Big) f(x(v+T))dv \\ &\quad + G(k+T, x(k+T), x(k+T-\tau(k+T)))\Big] e^{-\int_{k+T}^{t+T} h(u)du} dk \\ &= \big(1 - e^{-\int_{t-T}^{t} h(v)dv}\big)^{-1} \int_{t-T}^{t} \Big[- h(k)Q(k, x(k-\tau(k))) \\ &\quad + h(k)x(k) + \int_{-\infty}^{k} \Big(\sum_{j=1}^{n} g_j(k,v)\Big) f(x(v))dv \\ &\quad + G(k, x(k), x(k-\tau(k)))\Big] e^{-\int_k^{t} h(v)dv} dk \\ &= (Hx)(t).\end{aligned}$$

That is, $H : P_T \to P_T$.
We next show that H is continuous. To this end, we let

$$\eta = \sup_{t\in[0,T]} \mid (1 - e^{-\int_0^T h(v)dv})^{-1} \mid, \ \rho = \sup_{t\in[0,T]} \mid h(t) \mid, \ \gamma = \sup_{t\in[t-T,t]} e^{-\int_s^t h(v)dv}. \tag{3.1}$$

Let $\varphi, \psi \in P_T$, and $M = \eta\gamma T\Big[\rho K_1 + \rho + K_5 K_4 + K_2 + K_3\Big]$. Given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{M}$ such that $\| \varphi - \psi \| < \delta$. Thus,

$$\begin{aligned} \| H\varphi - H\psi \| &\leq \eta\gamma \int_{t-T}^{t} \Big[\rho K_1 \| \varphi - \psi \| \\ &\quad + \rho \| \varphi - \psi \| + K_5 K_4 \| \varphi - \psi \| \\ &\quad + K_2 \| \varphi - \psi \| + K_3 \| \varphi - \psi \| \Big] dk \\ &= \eta\gamma T\Big[\rho K_1 + \rho + K_5 K_4 + K_2 + K_3\Big] \| \varphi - \psi \| \\ &\leq M \| \varphi - \psi \| < \epsilon. \end{aligned}$$

Therefore, H is continuous.

To show that H is compact, we consider the sequence of periodic functions $\varphi_n \in P_T$ and assume that the sequence is uniformly bounded. Let R be such that $\| \varphi_n \| \leq R$, for all $n \in \mathbb{N}$. Then

$$\begin{aligned} \| (H\varphi_n) \| &= \| \big(1 - e^{-\int_{t-T}^{t} h(u)du}\big)^{-1} \int_{t-T}^{t} \Big[- h(s)Q(s, \varphi_n(s - \tau(s))) \\ &\quad + h(s)\varphi_n(s) + \int_{-\infty}^{s} \Big(\sum_{j=1}^{n} g_j(s,u)\Big) f(\varphi_n(u))du \\ &\quad + G(s, \varphi_n(s), \varphi_n(s - \tau(s)))\Big] e^{-\int_s^t h(u)du} ds \| \\ &\leq \eta\gamma \int_{t-T}^{t} \Big[\rho\Big(\mid Q(s, \varphi_n(s - \tau(s))) - Q(t, 0) \mid + \mid Q(t, 0) \mid \Big) \end{aligned}$$

$$
\begin{aligned}
&+ \rho \parallel \varphi_n \parallel + \int_{-\infty}^{s} \mid \Big(\sum_{j=1}^{n} g_j(s,u) \Big) \mid K_4 \parallel \varphi_n \parallel du \\
&+ \mid G(s,\varphi_n(s),\varphi_n(s-\tau(s))) - G(s,0,0) \mid + \mid G(s,0,0) \mid \Big] ds \\
\leq \;& \eta\gamma \int_{t-T}^{t} \Big[\rho \Big(K_1 \parallel \varphi_n \parallel + \beta_1 \Big) \\
&+ \rho \parallel \varphi_n \parallel + \int_{-\infty}^{s} \mid \Big(\sum_{j=1}^{n} g_j(s,u) \Big) \mid K_4 \parallel \varphi_n \parallel du \\
&+ K_2 \parallel \varphi_n \parallel + K_3 \parallel \varphi_n \parallel + \beta_2 \Big] ds \\
\leq \;& \eta\gamma T \Big[\rho \Big(K_1 R + \beta_1 \Big) + \rho R + K_5 K_4 R + K_2 R + K_3 R + \beta_2 \Big] := D,
\end{aligned}
$$

where $\beta_1 = \sup_{t\in[0,T]} \mid Q(t,0) \mid$, and $\beta_2 = \sup_{t\in[0,T]} \mid G(t,0,0) \mid$. Thus, the sequence $H\varphi_n$ is uniformly bounded. Differentiating $H\varphi_n$ gives

$$
\begin{aligned}
(H\varphi_n)'(t) \;=\; & -h(t)(H\varphi_n)(t) - h(t)Q(t,\varphi_n(t-\tau(t))) \\
&+ h(t)\varphi_n(t) + \int_{-\infty}^{t} \Big(\sum_{j=1}^{n} g_j(t,u) \Big) f(\varphi_n(u)) du \\
&+ G(t,\varphi_n(t),\varphi_n(t-\tau(t))).
\end{aligned}
$$

Consequently,

$$
\mid (H\varphi_n)'(t) \mid \;\leq\; \rho D + \rho \Big(K_1 R + \beta_1 \Big) + \rho R + K_5 K_4 R + K_2 R + K_3 R + \beta_2 := F,
$$

for all n. Thus the sequence $\{H\varphi_n\}$ is uniformly bounded and equicontinuous. The Arzela-Ascoli Theorem implies that $\{H\varphi_{n_k}\}$ uniformly converges to a continuous T-periodic function φ^*. Hence H is compact. □

Lemma 3.2. *Let J be defined by* (2.9) *and*

$$
K_1 < 1. \tag{3.2}
$$

Then $J : P_T \to P_T$ is a contraction.

Proof. Trivially, $J : P_T \to P_T$. For $\varphi, \psi \in P_T$, we have

$$
\parallel J\varphi - J\psi \parallel \;\leq\; K_1 \parallel \varphi - \psi \parallel . \tag{3.3}
$$

Hence J defines a contraction mapping with contraction constant K_1. □

Theorem 3.3. *Let* $\beta_1 = \sup_{t\in[0,T]} \mid Q(t,0) \mid$, *and* $\beta_2 = \sup_{t\in[0,T]} \mid G(t,0,0) \mid$. *Let* $\eta, \rho,$ *and* γ *be given by* (3.1). *Suppose* (2.2)-(2.6) *hold and there exist an arbitrary continuous function* $h : \mathbb{R} \to \mathbb{R}$ *such that*

(2.2) also hold. Let G be a positive constant such $\| x \| \leq G$ for $x \in P_T$ and that the inequality

$$\eta\gamma T\Big[\rho\Big(K_1G+\beta_1\Big)+\rho G+ K_5K_4G+K_2G+K_3G+\beta_2\Big]+K_1G+\beta_1\leq G, \tag{3.4}$$

holds. Then (1.1) has a T-periodic solution in $\mathbb{M}=\{\varphi\in P_T : \| \varphi \| \leq G\}$.

Proof. In view of the fact that $\mathbb{M}=\{\varphi\in P_T : \| \varphi \| \leq G\}$, Lemma 3.1 implies that H is compact and continuous. Also, from Lemma 3.2, J is a contraction.

We next show that if $\varphi,\psi\in\mathbb{M}$ we have $\| H\varphi+J\psi \| \leq G$. Let $\varphi,\psi\in\mathbb{M}$, then we have that

$$\begin{aligned}
\| H\varphi+J\psi \| &\leq \big(1-e^{-\int_{t-T}^{t}h(u)du}\big)^{-1}\int_{t-T}^{t}\Big[-h(s)Q(s,\varphi(s-\tau(s)))\\
&\quad + h(s)\varphi(s)+\int_{-\infty}^{s}\Big(\sum_{j=1}^{n}g_j(s,u)\Big)f(\varphi(u))du\\
&\quad + G(s,\varphi(s),\varphi(s-\tau(s)))\Big]e^{-\int_s^t h(u)du}ds+Q(t,\psi(t-\tau(t))\\
&\leq \eta\gamma T\Big[\rho\Big(K_1G+\beta_1\Big)+(\rho+ K_5K_4+K_2+K_3)G+\beta_2\Big]\\
&\quad + K_1G+\beta_1\leq G.
\end{aligned}$$

Thus, all the conditions of Krasnoselskii Theorem are satisfied. Thus, there exist a fixed point z in $\mathbb{M}$ such that $z=Hz+Jz$. By Lemma 2.1, this fixed point is a solution of (1.1) which is T-periodic. This completes the proof. □

Theorem 3.4. *Suppose (2.2)-(2.6) hold and there exist an arbitrary continuous function $h:\mathbb{R}\to\mathbb{R}$ such that (2.2) also hold. Let η,ρ, and γ be given by (3.1). If*

$$K_1+T\eta\gamma\Big[\rho K_1+\rho+K_4K_5+K_2+K_3\Big]\leq 1, \tag{3.5}$$

then (1.1) has a unique $T-$periodic solution.

Proof. Define the mapping $A : P_T \to P_T$ by

$$\begin{aligned}(A\varphi)(t) &= Q(t,\varphi(t-\tau(t))) + \big(1-e^{-\int_{t-T}^{t} h(u)du}\big)^{-1} \\ &\times \int_{t-T}^{t} \Big[-h(s)Q(s,\varphi(s-\tau(s))) \\ &+ h(s)\varphi(s) \\ &+ \int_{-\infty}^{s} \Big(\sum_{j=1}^{n} g_j(s,u)\Big) f(\varphi(u))du + G(s,\varphi(s),\varphi(s-\tau(s)))\Big] e^{-\int_s^t h(u)du} ds \,.\end{aligned}$$

Then, for $\varphi, \psi \in P_T$ we have,

$$\begin{aligned}\| (A\varphi)-(A\psi) \| &\leq K_1 \| \varphi-\psi \| + \eta\gamma \int_{t-T}^{t} \Big[\rho K_1 \| \varphi-\psi \| \\ &+ \rho \| \varphi-\psi \| + K_4K_5 \| \varphi-\psi \| + K_2 \| \varphi-\psi \| \\ &+ K_3 \| \varphi-\psi \| \Big] ds \\ &\leq \Big(K_1 + T\eta\gamma\Big[\rho K_1 + \rho + K_4K_5 + K_2 + K_3\Big]\Big) \| \varphi-\psi \| \,.\end{aligned}$$

This completes the proof. □

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