Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 1735-0611

CJMS. **4**(1)(2015),31-41

Solitons And Periodic Solutions To The Generalized Zakharov-Kuznetsov Benjamin-Bona-Mahoney Equation

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Abstract. This paper studies the generalized version of the Zakharov-Kuznetsov Benjamin-Bona-Mahoney equation. The functional variable method as well as the simplest equation method are applied to obtain solitons and singular periodic solutions to the equation. There are several constraint conditions that are naturally revealed in order for these specialized type of solutions to exist. The results of this paper generalizes the previous results that are reported in earlier publications.

Keywords: Solitons; periodic solutions;integrability.

2010 Mathematics subject classification: 35Q53; 35Q80.

1. Introduction

The study of nonlinear evolution equations (NLEEs) is extremely important in the field of applied sciences [1-13]. NLEEs appear in various areas such as applied mathematics, theoretical physics, engineering sciences as well as in mathematical biology. The dynamics of soliton propagation through nonlinear optical fibers, shallow water waves along lake shores and ocean beaches, Davydov solitons in α-helix proteins are all part of the applications of NLEEs appearing in various applied fields. Therefore it is compelling to carry out advanced research so that one can dig deeper in order to extract elite results with unprecedented novelty.

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Received: 06 October 2014
Revised: 09 December 2014
Accepted: 23 February 2015

There are several aspects of NLEEs that are necessary to be known. This paper focuses on one of such important aspects. It is the search for exact 1-soliton solution. There are several tools that are available nowadays to retrieve exact soliton solutions. Functional variable method and simplest equation method are recent popular approaches that are going to be implemented in this paper in order to obtain soliton solutions to a particular NLEE. The Zakharov-Kuznetsov Benjamin-Bona-Mahoney (ZK-BBM) equation that will be studied with a generalized setting.

2. GOVERNING EQUATION

The generalized form of ZK-BBM($m,\ n$) equation that is going to be considered in this paper is

$$(q^m)_t + \alpha\,(q^m)_x + \beta\,(q^n)_x + \gamma\left\{(q^m)_{xt} + (q^m)_{yy}\right\}_x = 0. \tag{2.1}$$

For $m = 1$ Eq. (1) will be reduced to the ZK-BBM equation with power-law nonlinearity [1]

$$q_t + aq_x + b\,(q^n)_x + c\,(q_{xt} + q_{yy})_x = 0. \tag{2.2}$$

Equation (2.1) is in general not integrable by the classic integrator, namely the inverse scattering transform that is the nonlinear analogue of Fourier Transform. However, there are several modern integrating approaches that reveals the mystery. Two such methods will be implemented in this paper. These are accounted in details in next two sections.

3. FUNCTIONAL VARIABLE METHOD

The functional variable method, which is a direct and effective algebraic method for the computation of compactons, solitons, solitary patterns and periodic solutions, was first proposed by Zerarka et al. [2]. This method was further developed by several authors [3-6]. We now summarize the functional variable method, established by Zerarka et al. [2], the details of which can be found in [3-6] among many others. Consider a general NLEE in the form

$$P\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial t \partial x}, ...\right) = 0, \tag{3.1}$$

where P is a polynomial in u and its partial derivatives. Using a wave variable $\xi = B_1 x + B_2 y - vt + \xi_0$ so that

$$u(x, y, t) = U(\xi), \tag{3.2}$$

Eq. (3.1) can be converted to an ordinary differential equation (ODE) as

$$Q\left(U, U', U'', U''', ...\right) = 0, \tag{3.3}$$

where Q is a polynomial in $U = U(\xi)$ and prime denotes derivative with respect to ξ. If all terms contain derivatives, then Eq. (3.3) is integrated where integration constants are considered zeros.

Let us make a transformation in which the unknown function $U(\xi)$ is considered as a functional variable in the form

$$U_\xi = F(U) \tag{3.4}$$

and some successively derivatives of U are

$$U_{\xi\xi} = \frac{1}{2}(F^2)', \tag{3.5}$$

$$U_{\xi\xi\xi} = \frac{1}{2}(F^2)''\sqrt{F^2},$$

$$U_{\xi\xi\xi\xi} = \frac{1}{2}[(F^2)'''F^2 + (F^2)''(F^2)'],$$

where $' = \frac{d}{dU}$.

The ODE (3.3) can be reduced in terms of U, F and its derivatives upon using the expressions of Eq. (3.5) into Eq. (3.3) gives

$$R(U, F, F', F'', F''', ...) = 0. \tag{3.6}$$

The key idea of this particular form Eq. (3.6) is of special interest because it admits analytical solutions for a large class of nonlinear wave type equations. After integration, the Eq. (3.6) provides the expression of F, and this in turn together with Eq. (3.4) give the relevant solutions to the original problem.

3.1. APPLICATION TO ZK-BBM(m, n) EQUATION.

In this subsection, we apply the functional variable method to construct the exact 1-soliton solutions of the ZK-BBM(m, n)

$$(q^m)_t + a\,(q^m)_x + b\,(q^n)_x + c\left((q^m)_{xt} + (q^m)_{yy}\right)_x = 0. \tag{3.7}$$

Under the traveling wave transformation

$$q(x, y, t) = U(\xi),\ \xi = B_1 x + B_2 y - vt + \xi_0, \tag{3.8}$$

we have

$$(aB_1 - v)\,(U^m)' + bB_1\,(U^n)' + cB_1(B_2^2 - vB_1)\,(U^m)''' = 0. \tag{3.9}$$

Integrating Eq. (3.9) and neglecting constant of integration, we find

$$(aB_1 - v)U^m + bB_1 U^n + cB_1(B_2^2 - vB_1)\,(U^m)'' = 0. \tag{3.10}$$

Now, we use the transformation

$$U(\xi) = V^{\frac{1}{m}}(\xi), \tag{3.11}$$

that will reduce Eq. (3.10) into the ODE

$$(aB_1 - v)V + bB_1 V^{\frac{n}{m}} + cB_1(B_2^2 - vB_1)V'' = 0. \tag{3.12}$$

According to Eq. (3.5), we get from Eq. (3.12) the expression of the function $F(V)$ reads

$$F(V) = \sqrt{\frac{v - aB_1}{cB_1(B_2^2 - vB_1)}} V \sqrt{1 - \frac{2mbB_1}{(v - aB_1)(n+m)} V^{\frac{n-m}{m}}}. \tag{3.13}$$

After making the change of variables

$$Z = \frac{2mbB_1}{(v - aB_1)(n+m)} V^{\frac{n-m}{m}}, \tag{3.14}$$

and using the relation $V_\xi = F(V)$, the solution of the Eq. (3.12) is in the following form

$$V(\xi) = \left\{ \frac{(aB_1 - v)(n+m)}{2mbB_1} \operatorname{csch}^2 \left(\frac{n-m}{2m} \sqrt{\frac{v - aB_1}{cB_1(B_2^2 - vB_1)}} \xi \right) \right\}^{\frac{m}{n-m}}. \tag{3.15}$$

Using the transformation (3.11), we can obtain the following soliton solutions of Eq. (3.7):

$$q_1(x,y,t) = \tag{3.16}$$

$$\left\{ \frac{(aB_1 - v)(n+m)}{2mbB_1} \operatorname{csch}^2 \left(\frac{n-m}{2m} \sqrt{\frac{v - aB_1}{cB_1(B_2^2 - vB_1)}} (B_1 x + B_2 y - vt + \xi_0) \right) \right\}^{\frac{1}{n-m}}$$

and

$$q_2(x,y,t) = \tag{3.17}$$

$$\left\{ \frac{(v - aB_1)(n+m)}{2mbB_1} \operatorname{sech}^2 \left(\frac{n-m}{2m} \sqrt{\frac{v - aB_1}{cB_1(B_2^2 - vB_1)}} (B_1 x + B_2 y - vt + \xi_0) \right) \right\}^{\frac{1}{n-m}}$$

for

$$(v - aB_1) \left\{ cB_1 \left(B_2^2 - vB_1 \right) \right\} > 0.$$

It is easy to see that solutions (18) and (19) can reduce to singular periodic solutions as follows:

$$q_3(x,y,t) = \tag{3.18}$$

$$\left\{\frac{(v-aB_1)(n+m)}{2mbB_1}\sec^2\left(\frac{n-m}{2m}\sqrt{\frac{aB_1-v}{cB_1(B_2^2-vB_1)}}(B_1x+B_2y-vt+\xi_0)\right)\right\}^{\frac{1}{n-m}}$$

and

$$q_4(x,y,t)= \tag{3.19}$$

$$\left\{\frac{(v-aB_1)(n+m)}{2mbB_1}\csc^2\left(\frac{n-m}{2m}\sqrt{\frac{aB_1-v}{cB_1(B_2^2-vB_1)}}(B_1x+B_2y-vt+\xi_0)\right)\right\}^{\frac{1}{n-m}}$$

for

$$(v-aB_1)\left\{cB_1\left(B_2^2-vB_1\right)\right\}<0.$$

These solutions given by (3.16)-(3.19) remain valid as long as $m \neq n$.

4. SIMPLEST EQUATION METHOD

Recently, a very powerful mathematical method for finding new exact solutions of NLEEs has been proposed by Kudryashov in [9, 10], called the simplest equation method. This useful method was successfully developed by Vitanov et al. in papers [11- 12] and the reference therein. Kudryashov used Weierstrass elliptic function as building blocks of exact solutions of number of differential equations. His idea is followed by using two types of differential equations i.e. Bernoulli and Riccati equations as building blocks of exact solution of classes of NLEEs [8-12].

The simplest equation method is illustrated in the following steps [8-12]:

Step-1: Suppose a NLEE with time-dependent coefficients is of the form

$$P\left(u,u_t,u_x,u_{tt},u_{xt},u_{xx},...\right)=0, \tag{4.1}$$

can be converted to the ODE

$$Q\left(U,U',U'',U''',...\right)=0, \tag{4.2}$$

via traveling wave variable $u(x,t)=U(\xi)$, $\xi=x-vt$, $U=U(\xi)$ is an unknown function, Q is a polynomial in the variable U and its derivatives. If all terms contain derivatives, then Eq. (4.2) is integrated while constants are considered zero.

Step-2: The basic idea of the simplest equation method, backs to express the solution $U(\xi)$ of Eq. (4.2) via truncated summation

$$U(\xi)=\sum_{l=0}^{N}a_l\left(z(\xi)\right)^l \tag{4.3}$$

in which the coefficients a_l are constants, independent of ξ and $z = z(\xi)$ is a function that satisfies class of ordinary differential equations. Such ordinary differential equations are called the simplest equations. Each simplest equation is characterized by the fact that has order lower than Eq. (4.2), and the general solution of that is known (or the way of finding its general solution is known, or at least some particular solutions of that is known). This means that the exact solution $U(\xi)$ of Eq. (4.2) can be presented by finite sum (4.3) with respect to the general solution $z = z(\xi)$ of the simplest equations.

As examples of simplest equations used in the literature, we can cite the Riccati equation, Jacobi elliptic function equation and Weierstrass elliptic function. In the present paper, the Riccati equation is used as simplest equation

$$\frac{dz}{d\xi} = k + az(\xi) + b\left(z(\xi)\right)^2, \tag{4.4}$$

where k, a and b are independent on ξ. When $k = 0$ and $a, b \neq 0$, the Bernoulli equation is acquired as;

$$\frac{dz}{d\xi} = az(\xi) + b\left(z(\xi)\right)^2. \tag{4.5}$$

It was found that Bernoulli equation leads to new traveling-wave and wave-front solution of Eq. (4.1). Eq. (4.5) admits the following exact solution

$$z(\xi) = \frac{a\exp\left(a(\xi+\xi_0)\right)}{1 - b\exp\left(a(\xi+\xi_0)\right)}, \tag{4.6}$$

for the case $a > 0, b < 0$, and

$$z(\xi) = -\frac{a\exp\left(a(\xi+\xi_0)\right)}{1 + b\exp\left(a(\xi+\xi_0)\right)}, \tag{4.7}$$

for the case $a < 0, b > 0$, where ξ_0 is a constant.
When $k = B \neq 0$ and $a = 0,\ b = A \neq 0$ the Riccati equation is obtained

$$\frac{dz}{d\xi} = B + A\left(z(\xi)\right)^2, \tag{4.8}$$

Eq. (4.8) admits the following exact solutions [13]:

$$z(\xi) = -\frac{\sqrt{-AB}}{A}\tanh\left(\sqrt{-AB}(\xi+\xi_0)\right), \tag{4.9}$$

$$z(\xi) = -\frac{\sqrt{-AB}}{A}\coth\left(\sqrt{-AB}(\xi+\xi_0)\right),$$

for $AB < 0$, and

$$z(\xi) = \frac{\sqrt{AB}}{A} \tan\left(\sqrt{AB}(\xi + \xi_0)\right), \tag{4.10}$$

$$z(\xi) = -\frac{\sqrt{AB}}{A} \cot\left(\sqrt{AB}(\xi + \xi_0)\right),$$

for $AB > 0$.

Step-3: Substituting (4.3) along with (4.5) into Eq. (4.2), the left hand side of Eq. (4.2) is converted into a polynomial in $z(\xi)$ and equating each coefficient of the polynomial to zero yields to a set of algebraic equations for $a_l,\ a,\ b,\ v$.

Step-4: Solving the algebraic equations obtained in step 3, and substituting the results into (4.3), the exact traveling wave solution for Eq. (4.1) is acquired.

Remark-1: N is a positive integer, in most cases, it is determined in prior. To determine the parameter N, usually the highest order of linear terms in the resulting equation are balanced with the highest order of nonlinear terms.

Remark-2: In Eq. (4.5), when $a = A$ and $b = -1$ the Bernoulli equation is obtained as

$$\frac{dz}{d\xi} = Az(\xi) - (z(\xi))^2. \tag{4.11}$$

Eq. (4.11) admits the following exact solution;

$$z(\xi) = \frac{A}{2}\left\{1 + \tanh\left(\frac{A}{2}(\xi + \xi_0)\right)\right\}, \tag{4.12}$$

for $A > 0$, and

$$z(\xi) = \frac{A}{2}\left\{1 - \tanh\left(\frac{A}{2}(\xi + \xi_0)\right)\right\}, \tag{4.13}$$

for $A < 0$.

4.1. APPLICATION TO ZK-BBM$(m,\ n)$ EQUATION.

In this subsection, we apply the simplest equation method to construct the exact 1-soliton solutions of the ZK-BBM$(m,\ n)$ equation

$$(q^m)_t + \alpha\,(q^m)_x + \beta\,(q^n)_x + \gamma\left((q^m)_{xt} + (q^m)_{yy}\right)_x = 0. \tag{4.14}$$

Applying transformation

$$q(x,y,t)=U(\xi),\ \xi=x+y-vt, \tag{4.15}$$

to Eq. (4.14) and integrating the resultant equation once leads to

$$(\alpha-v)\,U^m+\beta U^n+\gamma\,(1-v)\,(U^m)''=0. \tag{4.16}$$

Balancing $(U^m)''$ with U^n in Eq. (4.16) gives

$$(m-1)N+N+2=nN \Leftrightarrow mN-N+N+2=nN \Leftrightarrow N=\frac{2}{n-m},\ n\neq m.$$

To obtain the analytic solution, the transformation $U=V^{\frac{1}{n-m}}$ is applied in Eq. (4.16), and gives

$$\begin{aligned}&(\alpha-v)\,(n-m)^2V^2+\beta(n-m)^2V^3+\gamma\,(1-v)\\&\times\big(m(2m-n)(V')^2+m(n-m)VV''\big)=0.\end{aligned} \tag{4.17}$$

As mentioned before we suppose that the solution of Eq. (4.17) can be expressed by a polynomial in z as follows:

$$V(\xi)=\sum_{l=0}^{N}a_l\,(z(\xi))^l \tag{4.18}$$

where a_l are some functions of t to be determined, N is a positive integer which can be determined by balancing the highest order derivative term with the highest order nonlinear term after substituting ansatz (4.18) into Eq. (4.17), where z satisfies Eq. (4.5).
Balancing the order of VV'' and V^3 in Eq. (4.17), we have

$$N+N+2=3N\Leftrightarrow N=2.$$

Therefore; Eq. (4.18) can be rewritten as

$$V(\xi)=a_0+a_1z(\xi)+a_2\,(z(\xi))^2\,, \tag{4.19}$$

where $a_0,\ a_1$ and a_2 are some functions of t to be determined.
Substituting ansatz (4.19) along with Eq. (4.5) in Eq. (4.17) and setting all coefficients of powers of z to zero, a system of nonlinear algebraic equations is achieved and it's solution leads to

$$\begin{aligned}a&=\sqrt{\frac{2\gamma b^2m(m+n)(\alpha-1)-\beta a_2(m-n)^2}{m^2\gamma\beta a_2}},\\a_0&=0,\\a_1&=\sqrt{\frac{2\gamma b^2a_2m(m+n)(\alpha-1)-\beta a_2^2(m-n)^2}{m^2\gamma\beta b^2}},\\v&=\frac{2\gamma b^2m(m+n)+\beta a_2(n-m)^2}{2\gamma b^2m(m+n)}\end{aligned} \tag{4.20}$$

where a_2 and b are arbitrary constants.
Assuming $a > 0$ and choosing $b < 0$. Therefore, using solution (4.6) of Eq. (4.5), ansatz (4.19), we obtain the following exact solution of Eq. (4.17)

$$V(\xi) = \frac{K\exp\left(\sqrt{\frac{2\gamma b^2 m(m+n)(\alpha-1)-\beta a_2(m-n)^2}{m^2\gamma\beta a_2}}(\xi+\xi_0)\right)}{\left(1-b\exp\left(\sqrt{\frac{2\gamma b^2 m(m+n)(\alpha-1)-\beta a_2(m-n)^2}{m^2\gamma\beta a_2}}(\xi+\xi_0)\right)\right)^2}, \quad (4.21)$$

where

$$K = \frac{2\gamma b^2 m(m+n)(\alpha-1)-\beta a_2(m-n)^2}{m^2\gamma\beta b}.$$

Then the exact traveling-wave solution to Eq. (4.14) can be written as

$$q_1(x,y,t) = \frac{S\exp\left(\frac{1}{n-m}\sqrt{\frac{2\gamma b^2 m(m+n)(\alpha-1)-\beta a_2(m-n)^2}{m^2\gamma\beta a_2}}\,(X)\right)}{\left(1-b\exp\left(\sqrt{\frac{2\gamma b^2 m(m+n)(\alpha-1)-_2(m-n)^2}{m^2\gamma\beta a_2}}\,(X)\right)\right)^{\frac{2}{n-m}}}, \quad (4.22)$$

where

$$S = \left\{\frac{2\gamma b^2 m(m+n)(\alpha-1)-\beta a_2(m-n)^2}{m^2\gamma\beta b}\right\}^{\frac{1}{n-m}}.$$

$$X = x + y - \left(\frac{2\gamma b^2 m(m+n)+\beta a_2(n-m)^2}{2\gamma b^2 m(m+n)}\right)t + \xi_0$$

Substituting (4.19) along with Eq. (4.8) in Eq. (4.17) and setting all the coefficients of powers of z to zero, leads to a system of nonlinear algebraic equations, by solving it, we obtain

$$\begin{aligned} B &= -\frac{2\gamma A^2 m(m+n)(\alpha-1)-\beta a_2(m-n)^2}{4Am^2\gamma\beta a_2}, \\ a_0 &= -\frac{2\gamma A^2 a_2 m(m+n)(\alpha-1)-\beta a_2^2(m-n)^2}{4m^2\gamma\beta A^2}, \\ a_1 &= 0, \\ v &= \frac{2\gamma A^2 m(m+n)+\beta a_2(n-m)^2}{2\gamma A^2 m(m+n)} \end{aligned} \quad (4.23)$$

where A and a_2 are arbitrary constants.
Thus, using solutions (4.9)-(4.10) of Eq. (4.8) and ansatz (4.19), we obtain the exact solution of Eq. (4.17) and then the exact solution for the ZK-BBM(m, n) equation can be written as:

(1) Soliton solution:

$$q_2(x,y,t)=\frac{H}{\cosh^{\frac{2}{n-m}}\left(\sqrt{\frac{2\gamma A^2 m(m+n)(\alpha-1)-\beta a_2(m-n)^2}{4m^2\gamma\beta a_2}}\right)} \times\frac{1}{\left(x+y-\left(\frac{2\gamma A^2 m(m+n)+\beta a_2(n-m)^2}{2\gamma A^2 m(m+n)}\right)t+\xi_0\right)}, \tag{4.24}$$

where

$$H=\left\{-\frac{2\gamma A^2 m(m+n)(\alpha-1)-\beta a_2(m-n)^2}{4\gamma m^2 A^2\beta}\right\}^{\frac{1}{n-m}}.$$

(2) Singular soliton solution:

$$q_3(x,y,t)=\frac{H}{\sinh^{\frac{2}{n-m}}\left(\sqrt{\frac{2\gamma A^2 m(m+n)(\alpha-1)-\beta a_2(m-n)^2}{4m^2\gamma\beta a_2}}\right)} \times\frac{1}{\left(x+y-\left(\frac{2\gamma A^2 m(m+n)+\beta a_2(n-m)^2}{2\gamma A^2 m(m+n)}\right)t+\xi_0\right)}, \tag{4.25}$$

where

$$H=\left\{\frac{2\gamma A^2 m(m+n)(\alpha-1)-\beta a_2(m-n)^2}{4\gamma m^2 A^2\beta}\right\}^{\frac{1}{n-m}}.$$

(3) Periodic solutions:

$$q_4(x,y,t)=\frac{H}{\cos^{\frac{2}{n-m}}\left(\sqrt{\frac{\beta a_2(m-n)^2-2\gamma A^2 m(m+n)(\alpha-1)}{4m^2\gamma\beta a_2}}\right)} \times\frac{1}{\left(x+y-\left(\frac{2\gamma A^2 m(m+n)+\beta a_2(n-m)^2}{2\gamma A^2 m(m+n)}\right)t+\xi_0\right)}, \tag{4.26}$$

where

$$H=\left\{-\frac{2\gamma A^2 m(m+n)(\alpha-1)-\beta a_2(m-n)^2}{4\gamma m^2 A^2\beta}\right\}^{\frac{1}{n-m}}.$$

$$q_5(x,y,t) = \frac{H}{\sin^{\frac{2}{n-m}}\left(\sqrt{\frac{\beta a_2(m-n)^2 - 2\gamma A^2 m(m+n)(\alpha-1)}{4m^2\gamma\beta a_2}}\right)} \times \frac{1}{\left(x+y-\left(\frac{2\gamma A^2 m(m+n)+\beta a_2(n-m)^2}{2\gamma A^2 m(m+n)}\right)t+\xi_0\right)}, \tag{4.27}$$

where

$$H = \left\{-\frac{2\gamma A^2 m(m+n)(\alpha-1) - \beta a_2(m-n)^2}{4\gamma m^2 A^2 \beta}\right\}^{\frac{1}{n-m}}.$$

Once again, solutions (4.24)-(4.27) remain valid as long as $m \neq n$.

5. CONCLUSIONS

This paper studied the generalized version of ZK-BBM equation. Two integration techniques applied for finding the solitary waves and singular periodic solutions to the equation, that depends on the parameter regimes. These regimes are labeled as the constraint conditions in order for the solutions to exist. These results are going to be extremely useful in further researches.

There is a lot of prospect of further study in future. The perturbation terms will be added and the adiabatic parameter dynamics will be obtained for this equation. Additionally, the conservation laws needs to be computed for this equation. The Lie symmetry analysis will be employed to recover them. All of these will be reported in future.

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