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Closed Ideals, Point Derivations and Weak Amenability of Extended Little Lipschitz Algebras

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Abstract. Let (X, d) be a compact metric space and let K be a nonempty compact subset of X. Let $\alpha \in (0, 1]$ and let $\mathrm{Lip}(X, K, d^\alpha)$ denote the Banach algebra of all continuous complex-valued functions f on X for which $p_{(K,d^\alpha)}(f) = \sup\{\frac{|f(x)-f(y)|}{d^\alpha(x,y)} : x, y \in K, x \neq y\} < \infty$ when equipped with the algebra norm $||f||_{\mathrm{Lip}(X,K,d^\alpha)} = ||f||_X + p_{(K,d^\alpha)}(f)$, where $||f||_X = \sup\{|f(x)| : x \in X\}$. We denote by $\mathrm{lip}(X, K, d^\alpha)$ the closed subalgebra of $\mathrm{Lip}(X, K, d^\alpha)$ consisting of all $f \in \mathrm{Lip}(X, K, d^\alpha)$ for which $\frac{|f(x)-f(y)|}{d^\alpha(x,y)} \to 0$ as $d(x, y) \to 0$ with $x, y \in K$. In this paper we show that every proper closed ideal of $(\mathrm{lip}(X, K, d^\alpha), \|\cdot\|_{Lip(X,K,d^\alpha)})$ is the intersection of all maximal ideals containing it. We also prove that every continuous point derivation of $\mathrm{lip}(X, K, d^\alpha)$ is zero. Next we show that $\mathrm{lip}(X, K, d^\alpha)$ is weakly amenable if $\alpha \in (0, \frac{1}{2})$. We also prove that $\mathrm{lip}(\mathbb{T}, K, d^{\frac{1}{2}})$ is weakly amenable where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, d is the Euclidean metric on $\mathbb{T}$ and K is a nonempty compact set in $(\mathbb{T}, d)$.

Keywords: Banach function algebra, Extended Lipschitz algebra, Point derivation, Weak amenability.

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1. Introduction and preliminaries

We denote by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{T}$ and $\mathbb{N}$ the set of all real numbers, the set of all complex numbers, the unit circle in complex plane $\mathbb{C}$ and the set of all positive integer numbers, respectively.

Let A be a complex algebra and let φ be a multiplicative linear functional on A. A linear functional D on A is called a *point derivation* on A at φ if

$$D(fg) = \varphi(f)Dg + \varphi(g)Df,$$

for all $f, g \in A$. We say that φ is a *character* on A if $\varphi(f) \neq 0$ for some $f \in A$. We denote by $\ker(\varphi)$ the set of all $f \in A$ for which $\varphi(f) = 0$. Clearly, $\ker(\varphi)$ is a proper ideal of A.

Let A be a complex algebra. We denote by $\Delta(A)$ the set of all characters on A which is called the *character space* of A. For a subset S of $\Delta(A)$, we define $\ker(S) = A$ when $S = \emptyset$ and $\ker(S) = \bigcap_{\varphi \in S} \ker(\varphi)$ when $S \neq \emptyset$.

Let $(A, \|\cdot\|)$ be a commutative unital complex Banach algebra. We know that $\Delta(A) \neq \emptyset$ and it is a compact Hausdorff space with the Gelfand topology. Moreover, $\ker(\varphi)$ is a maximal ideal of A for all $\varphi \in \Delta(A)$ and every maximal ideal of A has the form $\ker(\psi)$ for some $\psi \in \Delta(A)$. Let I be an ideal of A. The *hull* of I is the set of all $\varphi \in \Delta(A)$ such that $\varphi(f) = 0$ for all $f \in I$. We denote by $\mathrm{hull}(I)$ the hull of I. Let S be a nonempty subset of $\Delta(A)$. We define

$$\begin{aligned} I_A(S) = \{f \in A \ \ : \ \ & \textit{there is an open set } V \textit{ in } \Delta(A) \textit{ with } S \subseteq V \\ & \textit{such that } \varphi(f) = 0 \textit{ for all } \varphi \in V\}, \end{aligned}$$

and $J_A(S) = \overline{I_A(S)}$, the closure of $I_A(S)$ in $(A, \|\cdot\|)$. Clearly, $\ker(S)$ and $J_A(S)$ are closed ideals of A and S is contained in $\mathrm{hull}(J_A(S))$.

Let $(A, \|\cdot\|)$ be a commutative unital complex Banach algebra. Then A is called *regular* if for every proper closed subset S of $\Delta(A)$ and each $\varphi \in \Delta(A) \setminus S$, there exists an f in A such that $\hat{f}(\varphi) = 1$ and $\hat{f}(S) = \{0\}$, where $\hat{f}$ is the Gelfand transform of f.

The following theorem is due to Šilov. For a proof see[10] or [6].

Theorem 1.1. *Let $(A, \|\cdot\|)$ be a regular Banach algebra. Then the following statements holds.*

(i) *If S is a nonempty closed subset of $\Delta(A)$, then*

$$\mathrm{hull}(J_A(S)) = S,$$

and if I is a closed ideal of A such that $\mathrm{hull}(I) = S$, *then $J_A(S)$ is contained in I.*

(ii) *If I is a closed ideal of A, then*

$$J_A(\mathrm{hull}(I)) \subseteq I \subseteq \ker(\mathrm{hull}(I)).$$

Let A be a complex algebra and $\mathfrak{X}$ be an A-module with respect to module operations $(a,x) \to x \cdot a : A \times \mathfrak{X} \to \mathfrak{X}$ and $(a,x) \to a \cdot x : A \times \mathfrak{X} \to \mathfrak{X}$. We say that $\mathfrak{X}$ is *symmetric* or *commutative* if $a \cdot x = x \cdot a$ for all $a \in A$ and $x \in \mathfrak{X}$. A complex linear map $D : A \to \mathfrak{X}$ is called an $\mathfrak{X}$-*derivation* on A if $D(ab) = Da \cdot b + a \cdot Db$ for all $a, b \in A$. For each $x \in \mathfrak{X}$, the map $\delta_x : A \to \mathfrak{X}$ defined by

$$\delta_x(a) = a \cdot x - x \cdot a \qquad (a \in A),$$

is an $\mathfrak{X}$-derivation on A. An $\mathfrak{X}$-derivation D on A is called *inner* $\mathfrak{X}$-derivation on A if $D = \delta_x$ for some $x \in X$.

Let A be a complex Banach algebra and $\mathfrak{X}$ be an A-module. We say that $\mathfrak{X}$ is a *Banach A-module* if $\mathfrak{X}$ is a Banach space and there exists a constant k such that

$$\|a \cdot x\| \leq k\|a\|\|x\|, \qquad \|x \cdot a\| \leq k\|a\|\|x\|,$$

for all $a \in A$ and $x \in \mathfrak{X}$.

Let $(A, \|\cdot\|)$ be a complex Banach algebra. Then A is a Banach A-module if the left and right module operations are considered by $a.b = ab$ and $a.b = ba$, respectively. If $\mathfrak{X}$ is a Banach A-module, then $\mathfrak{X}^*$, the dual space of $\mathfrak{X}$, is a Banach A-module with the natural module operations

$$(a \cdot \lambda)(x) = \lambda(x \cdot a), \; (\lambda \cdot a)(x) = \lambda(a \cdot x) \qquad (a \in A, \lambda \in \mathfrak{X}^*, x \in \mathfrak{X}).$$

Let A be a complex Banach algebra and $\mathfrak{X}$ be a Banach A-module. The set of all continuous $\mathfrak{X}$-derivations on A is a complex linear space, denoted by $Z^1(A, \mathfrak{X})$. The set of all inner $\mathfrak{X}$-derivations on A is a complex linear subspace of $Z^1(A, \mathfrak{X})$, denoted by $B^1(A, \mathfrak{X})$. The quotient space $Z^1(A, \mathfrak{X})/B^1(A, \mathfrak{X})$ is denoted by $H^1(A, \mathfrak{X})$ and called the *first cohomology group* of A with coefficients in $\mathfrak{X}$.

Definition 1.2. Let A be a complex Banach algebra. We say that A is *weakly amenable* if $H^1(A, A^*) = \{0\}$, that is, every continuous A^*-derivation on A is inner.

Above definition was first given by Johnson in [4]. The notion of weak amenability was first defined for commutative complex Banach algebras by Bade, Curtis and Dales in [2] as the following:

The commutative complex Banach algebra A is called *weakly amenable* if $Z^1(A, \mathfrak{X}) = \{0\}$ for every symmetric Banach A-module $\mathfrak{X}$, that is, every continuous $\mathfrak{X}$-derivation on A is necessarily inner for each symmetric Banach A-module $\mathfrak{X}$.

These two definitions are equivalent when A is commutative by [2, Theorem 1.5] and [5, Theorem 3.2].

It is known [2] that if A is a commutative unital complex Banach algebra and A has a nonzero continuous point derivation, then A is not weakly amenable.

The following result is useful and one can prove it as similar the proof of [3, Theorem VI.43.11].

Theorem 1.3. *Let A and B be complex Banach algebra, A be weakly amenable and $\Phi : A \to B$ be a continuous algebra homomorphism such that $\Phi(A)$ is dense in B. Then B is weakly amenable.*

Let X be a compact Hausdorff space. We denote by $C(X)$ the commutative unital complex Banach algebra consisting of all complex-valued continuous functions on X under the *uniform norm* on X which is defined by

$$\|f\|_X = \sup\{|f(x)| : x \in X\} \qquad (f \in C(X)).$$

A complex *Banach function algebra* on X is a complex subalgebra A of $C(X)$ such that A separates the points of X, contains 1_X (the constant function on X with value 1) and it is a unital Banach algebra under an algebra norm $\|\cdot\|$. Since $C(X)$ separates the points of X by Urysohn's lemma [7, Theorem 2.12], $1_X \in C(X)$ and $(C(X), \|\cdot\|_X)$ is a unital complex Banach algebra, we deduce that $(C(X), \|\cdot\|_X)$ is a Banach function algebra on X.

Let $(A, \|\cdot\|)$ be a Banach function algebra on X. For each $x \in X$, the map $e_x : A \to \mathbb{C}$, defined by $e_x(f) = f(x)$ $(f \in A)$, is an element of $\Delta(A)$ which is called the *evaluation character* on A at x. It follows that A is semisimple and $\|f\|_X \leq \|\hat{f}\|_{\Delta(A)}$ for all $f \in A$. Moreover, the map $E_X : X \to \Delta(A)$ defined by $E_X(x) = e_x$ is injective and continuous. If E_X is surjective, then we say that A is *natural*. In this case, E_X is a homeomorphism from X onto $\Delta(A)$. It is known that if $(A, \|\cdot\|)$ is a self-adjoint inverse-closed Banach function algebra on X then A is natural. Therefore, $(C(X), \|\cdot\|_X)$ is natural.

Let A be a complex Banach function algebra on a compact Hausdorff X. If A is regular, then for each proper closed subset F of X and each $x \in X \setminus F$ there exists a function f in A such that $f(x) = 1$ and $f(F) = \{0\}$. Moreover, the converse of above statement holds whenever A is natural.

Theorem 1.4. *Let X be a compact Hausdorff space, $(A.\|\cdot\|_A)$ be a natural Banach function algebra on X and $(B, \|\cdot\|_B)$ be a regular Banach function algebra on X such that $B \subseteq A$. Then $(A, \|\cdot\|_A)$ is regular.*

Proof. Let F be a proper closed subset of X and $x \in X \setminus E$. The regularity of $(B, \|\cdot\|_B)$ implies that there exists a function f in B such that $f(x) = 1$ and $f(F) = \{0\}$. Since $B \subseteq A$, and $(A, \|\cdot\|_A)$ is natural, the proof is complete. □

Let (X, d) be a metric space. For $\alpha > 0$, a complex-valued function f on X is a *Lipschitz function* of order α on X if there exists a positive constant M such that $|f(x)-f(y)| \leq M(d(x,y))^\alpha$ for all $x, y \in X$. If $\alpha \in (0,1]$, the map $d^\alpha : X \times X \to \mathbb{R}$, defined by $d^\alpha(x,y) = (d(x,y))^\alpha$ $(x, y \in X)$, is a metric on X and the induced topology on X by d^α coincides with the induced topology on X by d.

Let (X, d) be a compact metric space and $\alpha \in (0,1]$. We denote by $\mathrm{Lip}(X, d^\alpha)$ the set of all complex-valued Lipschitz functions on (X, d^α). Then $\mathrm{Lip}(X, d^\alpha)$ is a complex subalgebra of $C(X)$ and $1_X \in \mathrm{Lip}(\mathrm{X}, \mathrm{d}^\alpha)$. Moreover, $\mathrm{Lip}(X, d)$ separates the points of X. For a nonempty subset K of X, and a complex-valued function f on K, we set

$$p_{(K,d^\alpha)}(f) = \sup\{\frac{|f(x) - f(y)|}{d^\alpha(x,y)} : x, y \in K, x \neq y\}.$$

Clearly, $f \in \mathrm{Lip}(X, d^\alpha)$ if and only if $p_{(X,d^\alpha)}(f) < \infty$. The d^α-Lipschitz norm $\|\cdot\|_{\mathrm{Lip}(X,d^\alpha)}$ on $\mathrm{Lip}(X, d^\alpha)$ is defined by

$$\|f\|_{\mathrm{Lip}(X,d^\alpha)} = \|f\|_X + p_{(X,d^\alpha)}(f) \qquad (f \in \mathrm{Lip}(X, d^\alpha)).$$

Then $(\mathrm{Lip}(X, d^\alpha), \|\cdot\|_{\mathrm{Lip}(X,d^\alpha)})$ is a commutative unital complex Banach algebra. The set of all complex-valued functions f on X for which

$$\lim_{d(x,y)\to 0} \frac{|f(x) - f(y)|}{d^\alpha(x,y)} = 0,$$

is a closed complex subalgebra of $(\mathrm{Lip}(X, d^\alpha), \|\cdot\|_{\mathrm{Lip}(X,d^\alpha)})$ containing 1_X. This algebra is called *little Lipschitz algebra* of order α on (X, d) and denoted by $\mathrm{lip}(X, d^\alpha)$. It is known that $\mathrm{Lip}(X, d^\beta)$ is a complex subalgebra of $\mathrm{lip}(X, d^\alpha)$ whenever $0 < \alpha < \beta \leq 1$. Lipschitz algebras $\mathrm{Lip}(X, d^\alpha)$ and little Lipschitz algebras $\mathrm{lip}(X, d^\alpha)$ were first studied by Sherbert in [8] and [9].

Bade, Curtis and Dales studied the weak amenability of little Lipschitz algebras in [2] and obtained the following results that we use them in the sequel.

Theorem 1.5 (see [2, Theorem 3.10]). *Let (X, d) be a compact metric space and $\alpha \in (0, \frac{1}{2})$. Then $\mathrm{lip}(X, d^\alpha)$ is weakly amenable.*

Theorem 1.6 (see [2, Theorem 3.13]). *Let d be the Euclidean metric on $\mathbb{T}$. Then $\mathrm{lip}(\mathbb{T}, d^{\frac{1}{2}})$ is weakly amenable.*

Let (X, d) be a compact metric space, K be a nonempty compact subset of X and $\alpha \in (0,1]$. We denote by $\mathrm{Lip}(X, K, d^\alpha)$ the set of

$f \in C(X)$ for which $p_{(K,d^\alpha)}(f) < \infty$. In fact,

$$\mathrm{Lip}(X,K,d^\alpha) = \{f \in C(X) : f|_K \in \mathrm{Lip}(K,d^\alpha)\}.$$

Clearly, $\mathrm{Lip}(X,d^\alpha) \subseteq \mathrm{Lip}(X,K,d^\alpha)$ and $\mathrm{Lip}(X,K,d^\alpha) = \mathrm{Lip}(X,d^\alpha)$ if $K = X$. Moreover, $\mathrm{Lip}(X,K,d^\alpha)$ is a self-adjoint inverse-closed complex subalgebra of $C(X)$. It is easy to see that $\mathrm{Lip}(X,K,d^\alpha)$ is a complex subalgebra of $C(X)$ and a unital Banach algebra under the algebra norm $\|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)}$ defined by

$$\|f\|_{\mathrm{Lip}(X,K,d^\alpha)} = \|f\|_X + p_{(K,d^\alpha)}(f) \qquad (f \in \mathrm{Lip}(X,K,d^\alpha)).$$

Therefore, $(\mathrm{Lip}(X,K,d^\alpha), \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$ is a natural Banach function algebra on X. This algebra is called *extended Lipschitz algebra* of order α on (X,d) with respect to K. We denote by $\mathrm{lip}(X,K,d^\alpha)$ the set of all $f \in C(X)$ for which

$$\lim_{\substack{d(x,y)\to 0\\ x,y\in K}} \frac{|f(x)-f(y)|}{d^\alpha(x,y)} = 0.$$

In fact,

$$\mathrm{lip}(X,K,d^\alpha) = \{f \in C(X) : f|_K \in \mathrm{lip}(K,d^\alpha)\}.$$

Clearly, $\mathrm{lip}(X,K,d^\alpha)$ is a complex subalgebra of $\mathrm{Lip}(X,K,d^\alpha)$ containing 1_X and a closed set in $(\mathrm{Lip}(X,K,d^\alpha), \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$. This algebra is called *extended little Lischitz algebra* of order α on (X,d) with respect to K. Clearly, $\mathrm{Lip}(X,d) \subseteq \mathrm{lip}(X,d^\alpha) \subseteq \mathrm{lip}(X,K,d^\alpha)$ and $\mathrm{lip}(X,K,d^\alpha) = \mathrm{lip}(X,d^\alpha)$ if $K = X$. Moreover, $\mathrm{lip}(X,K,d^\alpha)$ is self-adjoint inverse-closed. Therefore, $(\mathrm{lip}(X,K,d^\alpha), \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$ is a natural Banach function algebra on X. It is clear that $\mathrm{Lip}(X,K,d^\alpha) = \mathrm{lip}(X,K,d^\alpha) = C(X)$ whenever K is finite.

Extended Lipschitz algebras and extended little Lipschitz algebras were first introduced in [4].

The following result is obtained in [1] that we use it in the sequel.

Theorem 1.7 (see [1, Corollary 2.9])**.** *Let (X,d) be a compact metric space, K be a nonempty compact subset of X and $\alpha \in (0,1)$. Then* $\mathrm{lip}(X,d^\alpha)$ *is dense in* $(\mathrm{lip}(X,K,d^\alpha), \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$.

Let (X,d) be a metric space, f be a real-valued function on X and $k > 0$. The real-valued function $T_k f$ on X defined by

$$(T_k f)(x) = \begin{cases} -k & f(x) < -k \\ f(x) & -k \leqslant f(x) \leqslant k \\ k & f(x) > k \end{cases} \qquad (x \in X)$$

is called the *truncation* of f at k.

The following result is useful in the sequel and its proof is straightforward.

Theorem 1.8. *Let (X, d) be a compact metric space, K be a nonempty compact subset of X and $\alpha \in (0, 1)$. Suppose that f is a real-valued function in* $\operatorname{lip}(X, K, d^\alpha)$ *and $k > 0$. Then $T_k f$ is in* $\operatorname{lip}(X, K, d^\alpha)$.

In Section 2, we show that extended little Lipschitz algebras are regular. We also determine the structure of closed ideals of these Banach algebras. In Section 3, we prove that every continuous point derivation on an extended little Lipschitz algebra is zero. In Section 4, we show that certain extended little Lipschitz algebras are weakly amenable.

2. Closed Ideals of Extended Little Lipschitz Algebras

Throughout this section we always assume that (X, d) is a compact metric space, K is a nonempty compact subset of X and $\alpha \in (0, 1)$. We show that every closed ideal I in $\operatorname{lip}(X, K, d^\alpha)$ is the form $\ker(E_X(Y))$ for some closed subset Y of X. Equivalently, every closed ideal in $\operatorname{lip}(X, K, d^\alpha)$ is the intersection of the maximal ideals containing it. To prove of this result we need following two results.

Theorem 2.1. *Let $A = \operatorname{lip}(X, K, d^\alpha)$. Then $(A, \|\cdot\|_{\operatorname{Lip}(X,K,d^\alpha)})$ is regular.*

Proof. We know that $(A, \|\cdot\|_{\operatorname{Lip}(X,K,d^\alpha)})$ is a natural Banach function algebra on X. On the other hand, $(\operatorname{Lip}(X, d^1), \|\cdot\|_{\operatorname{Lip}(X,d^1)})$ is a regular Banach function algebra on X by [9, Proposition 2.1]. Since $\operatorname{Lip}(X, d^1)$ is a subset of $\operatorname{lip}(X, K, d^\alpha)$, we deduce that $(A, \|\cdot\|_{\operatorname{Lip}(X,K,d^\alpha)})$ is regular by Theorem 1.4. □

Lemma 2.2. *Let $f \in \operatorname{lip}(X, K, d^\alpha)$ with $Z(f) \neq \emptyset$, where $Z(f) = \{x \in X : f(x) = 0\}$. Then there exists a sequence $\{f_n\}_{n=1}^\infty$ in* $\operatorname{lip}(X, K, d^\alpha)$ *satisfying:*

(i) *for each $n \in \mathbb{N}$, there is an open set U_n in X with $Z(f) \subseteq U_n$ such that $f_n|_{U_n} = f|_{U_n}$,*

(ii) $\lim_{n\to\infty} \|f_n\|_{\operatorname{Lip}(X,K,d^\alpha)} = 0$.

Proof. We first assume that f is real-valued. Let $n \in \mathbb{N}$. We define $f_n = T_{\frac{1}{n}} f$, the truncation of f at $\frac{1}{n}$. Then $f_n \in \operatorname{lip}(X, K, d^\alpha)$ by Theorem 1.8. Let

$$U_n = \{x \in X : |f(x)| < \frac{1}{n}\}.$$

The continuity of f implies that U_n is an open set in X. Moreover, $Z(f) \subseteq U_n$ and $f_n|_{U_n} = f|_{U_n}$. Hence (i) is satisfied.

Since $\|f_n\|_X \leq \|T_{\frac{1}{n}} f\|_X \leq \frac{1}{n}$ for each $n \in \mathbb{N}$, we deduce that

$$\lim_{n\to\infty} \|f_n\|_X = 0.$$

To establish (ii), it is remains to show that

$$\lim_{n\to\infty} p_{(K,d^\alpha)}(f_n) = 0.$$

Let $\varepsilon > 0$ be given. Since $f \in \mathrm{lip}(X, K, d^\alpha)$, there is a $\delta > 0$ such that

$$\frac{|f(x) - f(y)|}{d^\alpha(x, y)} < \frac{\varepsilon}{2},$$

whenever $x, y \in K$ and $0 < d(x, y) < \delta$.

Let $x, y \in K$ with $0 < d(x, y) < \delta$. By definition of $\{f_n\}_{n=1}^\infty$, we have $|f_n(x) - f_n(y)| \leq |f(x) - f(y)|$ for all $n \in \mathbb{N}$ and so

$$\frac{|f_n(x) - f_n(y)|}{d^\alpha(x, y)} \leq \frac{|f(x) - f(y)|}{d^\alpha(x, y)} < \frac{\varepsilon}{2},$$

for all $n \in \mathbb{N}$.

Let $x, y \in K$ with $d(x, y) \geq \delta$. Then we have

$$\frac{|f_n(x) - f_n(y)|}{d^\alpha(x, y)} \leq \frac{|f_n(x)| + |f_n(y)|}{\delta^\alpha} \leq \frac{2}{n\delta^\alpha},$$

for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ with $n > \frac{4}{3\delta^\alpha}$. Then we have

$$\frac{|f_n(x) - f_n(y)|}{d^\alpha(x, y)} < \frac{\varepsilon}{2},$$

for all $x, y \in K$ with $x \neq y$. Thus

$$p_{(K,d^\alpha)}(f_n) \leq \frac{\varepsilon}{2} < \varepsilon,$$

and so

$$\lim_{n\to\infty} p_{(K,d^\alpha)}(f_n) = 0.$$

Therefore, (ii) holds.

We now assume that f is complex-valued. Let $g = \mathrm{Re}\, f$ and $h = \mathrm{Im}\, f$. Then g and h are real-valued functions in $\mathrm{lip}(X, K, d^\alpha)$, $f = g + ih$ and $Z(f) = Z(g) \bigcap Z(h)$. By the above argument, there exists the sequence of real-valued functions $\{g_n\}_{n=1}^\infty$ and $\{h_n\}_{n=1}^\infty$ in $\mathrm{lip}(X, K, d^\alpha)$ satisfying:

- (I) for each $n \in \mathbb{N}$, there is an open set V_n in X with $Z(h) \subseteq V_n$ such that $g_n|_{V_n} = g|_{V_n}$,
- (II) $\lim_{n\to\infty} \|g_n\|_{\mathrm{Lip}(X,K,d^\alpha)} = 0$,
- (III) for each $n \in \mathbb{N}$, there is an open set W_n in X with $Z(h) \subseteq W_n$ such that $h_n|_{W_n} = h|_{W_n}$,
- (IV) $\lim_{n\to\infty} \|h_n\|_{\mathrm{Lip}(X,K,d^\alpha)} = 0$.

Let $f_n = g_n + ih_n$ and $U_n = V_n \bigcap W_n$ for all $n \in \mathbb{N}$. Then $\{f_n\}_{n=1}^\infty$ is a sequence in $\mathrm{lip}(X, K, d^\alpha)$ and by (II) and (IV) we have

$$\lim_{n\to\infty} \|f_n\|_{\mathrm{Lip}(X,K,d^\alpha)} = 0.$$

Moreover, U_n is an open set in X with $Z(f) \subseteq U_n$ and $f_n|_{U_n} = f|_{U_n}$ for all $n \in \mathbb{N}$ by (I) and (III). Therefore, the sequence $\{f_n\}_{n=1}^{\infty}$ satisfying (i) and (ii). □

Theorem 2.3. *Let $A = \mathrm{lip}(X, K, d^{\alpha})$ and I be a closed ideal of $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^{\alpha})})$. Then $I = \ker(\mathrm{hull}(I))$.*

Proof. If $I = A$, then $\mathrm{hull}(I) = \emptyset$ and so

$$\ker(\mathrm{hull}(I)) = A = I.$$

Let I be a proper closed ideal of $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^{\alpha})})$. By Theorem 2.1, $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^{\alpha})})$ is a regular commutative Banach algebra. Therefore, by part (ii) of Theorem 1.1 we have

$$J_A(\mathrm{hull}(I)) \subseteq I \subseteq \ker(\mathrm{hull}(I)). \tag{2.1}$$

Since $\mathrm{hull}(I) \neq \emptyset$, we deduce that $\ker(\mathrm{hull}(I))$ is a proper closed ideal of $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^{\alpha})})$. Let $f \in \ker(\mathrm{hull}(I))$. Then $Z(f) \neq \emptyset$. By Lemma 2.2, there exists a sequence $\{f_n\}_{n=1}^{\infty}$ in A satisfying:

(i) for each $n \in \mathbb{N}$, there is an open set U_n in X with $Z(f) \subseteq U_n$ such that $f_n|_{U_n} = f|_{U_n}$,

(ii) $\lim\limits_{n\to\infty} ||f_n||_{\mathrm{Lip}(X,K,d^{\alpha})} = 0$.

Let $n \in \mathbb{N}$. Define $V_n = \{e_x : x \in U_n\}$. Then V_n is an open set in $\Delta(\mathrm{lip}(X, K, d^{\alpha}))$ and we have

$$\hat{f_n}(e_x) = e_x(f_n) = f_n(x) = f(x) = e_x(f) = \hat{f}(e_x),$$

for all $x \in U_n$. So $\hat{f_n}(\varphi) = \hat{f}(\varphi)$ for all $\varphi \in V_n$. This implies that $f \in J_A(\mathrm{hull}(I))$ by regularity of $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^{\alpha})})$ and [9, Lemma 3.1]. So

$$\ker(\mathrm{hull}(I)) \subseteq J_A(\mathrm{hull}(I)). \tag{2.2}$$

From (2.1) and (2.2), we have $I = \ker(\mathrm{hull}(I))$. □

Corollary 2.4. *Let $A = \mathrm{lip}(X, K, d^{\alpha})$. and I be a proper ideal of A. Then the following statements are equivalent.*

(i) *I is a closed ideal of $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^{\alpha})})$.*

(ii) *There exists a closed subset Y of X such that $I = \ker(E_X(Y))$.*

(iii) *I is the intersection of all maximal ideals of A containing it.*

Proof. (i)⇒(ii). By Theorem 2.3, we have

$$I = \ker(\mathrm{hull}(I)). \tag{2.3}$$

Since $\mathrm{hull}(I) = \{\varphi \in \Delta(A) : \varphi(f) = 0 \ (f \in I)\}$ and $E_X : X \to \Delta(A)$ defined by $E_X(x) = e_x \ \ (x \in X)$ is bijective, we have

$$\mathrm{hull}(I) = \{E_X(x) : x \in X, \ f(x) = 0 \ (f \in I)\}.$$

Let $Y = \{x \in X : f(x) = 0 \ (f \in I)\}$. Then Y is a closed set in (X, d) and $\mathrm{hull}(I) = E_X(Y)$. Therefore, $I = \ker(E_X(Y))$ by (2.3) and so (ii) holds.
(ii)⇒(iii). Suppose that Y is a closed set in (X, d) such that

$$I = \ker(E_X(Y)). \tag{2.4}$$

Since

$$\ker(E_X(Y)) = \bigcap_{\varphi \in E_X(Y)} \{f \in A : \varphi(f) = 0\}$$

and

$$E_X(Y) = \{e_y : y \in Y\},$$

we deduce that

$$\ker(E_X(Y)) = \bigcap_{y \in Y} \{f \in A : f(y) = 0\}. \tag{2.5}$$

From (2.4) and (2.5), we have

$$I = \bigcap_{y \in Y} \{f \in A : f(y) = 0\}. \tag{2.6}$$

Since $\{f \in A : f(y) = 0\}$ is a maximal ideal of A for all $y \in Y$, we deduce that I contains the intersection of all maximal ideals M of A such that $I \subseteq M$.

On the other hand, I is contained in the intersection of all maximal ideals M of A such that $I \subseteq M$. Therefore, (iii) holds.
(iii)⇒(ii). It is obvious. □

3. Point Derivations of Extended Little Lipschitz Algebras

Let $(A, \|\cdot\|)$ be a commutative complex unital Banach algebra and I be an ideal of A. Set

$$I^2 = \{\sum_{i=1}^{n} \alpha_i f_i g_i : n \in \mathbb{N}, \alpha_i \in \mathbb{C}, \ f_i, g_i \in I \ (i \in \{1, ..., n\})\}.$$

Clearly, I^2 is also an ideal of A. For each $\varphi \in \Delta(A)$, we denote by $\mathfrak{D}_\varphi$ the set of all continuous point derivations on A at φ. We say that $\mathfrak{D}_\varphi$ is *nontrivial* if $\mathfrak{D}_\varphi \setminus \{0\} \neq \emptyset$.

Let (X, d) be a compact metric space, K be a nonempty compact subset of X and $\alpha \in (0, 1)$. We show that $\mathfrak{D}_\varphi = \{0\}$ for all $\varphi \in \Delta(\mathrm{lip}(X, K, d^\alpha))$. To prove this fact, we need the following theorem is due to Singer and Wermer obtained in [11].

Theorem 3.1. *Let $(A, \|\cdot\|)$ be a commutative complex unital Banach algebra with unit 1 and $\varphi \in (A)$. Then*

(i) $D \in A^*$ *is a point derivation at* φ *if and only if* $D1 = 0$ *and* $Df = 0$ *for all* $f \in \overline{(\ker(\varphi))^2}$.

(ii) $\mathfrak{D}_\varphi$ *is nontrivial if and only if* $\ker(\varphi) \neq \overline{(\ker(\varphi))^2}$.

Theorem 3.2. *Let* (X, d) *be a compact metric space,* K *be a nonempty compact subset of* X*,* $\alpha \in (0, 1)$ *and* $A = \mathrm{lip}(X, K, d^\alpha)$*. If* $\varphi \in \Delta(A)$*, then*

$$\overline{(\ker(\varphi))^2} = \ker(\varphi).$$

Proof. Let $\varphi \in \Delta(A)$. Clearly,

$$(\ker(\varphi))^2 \subseteq \ker(\varphi).$$

Since $\ker(\varphi)$ is closed in $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$, we deduce that

$$\overline{(\ker(\varphi))^2} \subseteq \ker(\varphi). \tag{3.1}$$

Since $\ker(\varphi)$ is a proper ideal of A and $\overline{(\ker(\varphi))^2}$ is an closed ideal of $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$, we conclude that $\overline{(\ker(\varphi))^2}$ is a proper closed ideal of $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$ by (3.1). By Corollary 2.4, $\overline{(\ker(\varphi))^2}$ is the intersection of all maximal ideals of A containing it. The naturality of A implies that there exists a unique $x \in X$ such that $\varphi = e_x$. Let M be a maximal ideal of A such that

$$\overline{(\ker(\varphi))^2} \subseteq M. \tag{3.2}$$

Since $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$ is a unital commutative Banach algebra, there exists a unique $\psi \in \Delta(A)$ such that $M = \ker(\psi)$. The naturality of Banach function algebra $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$ implies that there exists a unique $y \in X$ such that $\psi = e_y$. We claim that $y = x$. Let $y \neq x$. Since $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$ is a regular Banach function algebra on X, there exists a function $f \in A$ such that $f(x) = 0$ and $f(y) = 1$. So $f^2 \in (\ker(\varphi))^2$ and $f^2 \notin M$, contradicting to (3.2). Hence, our claim is justified. This implies that $M = \ker(e_x) = \ker(\varphi)$. Hence, $\ker(\varphi)$ is the only maximal ideal space of A containing $\overline{(\ker(\varphi))^2}$. Therefore, by Corollary 2.4 we have

$$\overline{(\ker(\varphi))^2} = \ker(\varphi).$$

This completes the proof. □

Theorem 3.3. *Let* (X, d) *be a compact metric space,* K *be a nonempty compact subset of* X*,* $\alpha \in (0, 1)$ *and* $A = \mathrm{lip}(X, K, d^\alpha)$*. Then every continuous point derivation on* $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$ *is zero.*

Proof. Let $\varphi \in \Delta(A)$. By Theorem 3.2, we have

$$\overline{(\ker(\varphi))^2} = \ker(\varphi).$$

This implies that $\mathfrak{D}_\varphi = \{0\}$ by Theorem 3.1. Hence, the proof is complete. □

4. Weak Amenability of Certain Extended Little Lipschitz Algebras

Let (X, d) be a compact metric space, K be a nonempty compact subset of X, $\alpha \in (0, 1)$ and $A = \mathrm{lip}(X, K, d^\alpha)$. By Theorem 3.3, all continuous point derivations on $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$ are zero. It follows that $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$ has some chance of being weakly amenable. We give some sufficient conditions that $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$ to be weakly amenable.

Theorem 4.1. *Let (X, d) be a compact metric space, K be a nonempty compact subset of X, $\alpha \in (0, \frac{1}{2})$ and $A = \mathrm{lip}(X, K, d^\alpha)$. Then $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$ is weakly amenable.*

Proof. By Theorem 1.5, $(\mathrm{lip}(X, d^\alpha), \|\cdot\|_{\mathrm{Lip}(X,d^\alpha})$ is weakly amenable. Let $\Phi : \mathrm{lip}(X, d^\alpha) \to A$ defined by $\Phi(f) = f$ $(f \in \mathrm{lip}(X, d^\alpha))$. Clearly, $\Phi(\mathrm{lip}(X, d^\alpha)) = \mathrm{lip}(X, d^\alpha)$ and Φ is an algebra homomorphism from $\mathrm{lip}(X, d^\alpha)$ into A. Since

$$\begin{aligned} \|\Phi(f)\|_{\mathrm{Lip}(X,K,d^\alpha)} &= \|f\|_{\mathrm{Lip}(X,K,d^\alpha)} = \|f\|_X + p_{(K,d^\alpha)}(f) \\ &\leq \|f\|_X + p_{(X,d^\alpha)}(f) = \|f\|_{\mathrm{Lip}(X,d^\alpha)} \end{aligned}$$

for all $f \in \mathrm{lip}(X, d^\alpha)$, we deduce that Φ is continuous. On the other hand, $\mathrm{lip}(X, d^\alpha)$ and so $\Phi(\mathrm{lip}(X, d^\alpha))$ is dense in $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$ by Theorem 1.7. Therefore, $(A, \|\cdot\|_{\mathrm{Lip}(X,K,d^\alpha)})$ is weakly amenable by Theorem 1.3. □

Theorem 4.2. *Let d be the Euclidean metric on $\mathbb{T}$, K be a nonempty compact set in $(\mathbb{T}, d)$ and $\alpha \in (0, \frac{1}{2}]$. Then $(\mathrm{lip}(\mathbb{T}, K, d^\alpha), \|\cdot\|_{\mathrm{Lip}(\mathbb{T},K,d^\alpha)})$ is weakly amenable.*

Proof. By Theorem 4.1, $(\mathrm{lip}(\mathbb{T}, K, d^\alpha), \|\cdot\|_{\mathrm{Lip}(\mathbb{T},d^\alpha)})$ is weakly amenable whenever $\alpha \in (0, \frac{1}{2})$. Suppose that $\alpha = \frac{1}{2}$. By Theorem 1.6, $(\mathrm{lip}(\mathbb{T}, d^\alpha), \|\cdot\|_{\mathrm{Lip}(\mathbb{T},d^\alpha)})$ is weakly amenable. By given argument in the proof of Theorem 4.1, the map $\Phi : \mathrm{lip}(\mathbb{T}, d^\alpha) \to \mathrm{lip}(\mathbb{T}, K, d^\alpha)$ defined by $\Phi(f) = f$ is a continuous algebra homomorphism from $(\mathrm{lip}(\mathbb{T}, d^\alpha), \|\cdot\|_{\mathrm{Lip}(\mathbb{T},d^\alpha)})$ into $(\mathrm{lip}(\mathbb{T}, K, d^\alpha), \|\cdot\|_{\mathrm{Lip}(\mathbb{T},K,d^\alpha)})$ and $\Phi(\mathrm{lip}(\mathbb{T}, d^\alpha))$ is dense in $(\mathrm{lip}(\mathbb{T}, K, d^\alpha), \|\cdot\|_{\mathrm{Lip}(\mathbb{T},K,d^\alpha)})$. Therefore, $(\mathrm{lip}(\mathbb{T}, K, d^\alpha), \|\cdot\|_{\mathrm{Lip}(\mathbb{T},K,d^\alpha)})$ is weakly amenable by Theorem 1.3. □

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