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The sum of two maximal monotone operator is of type FPV

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Abstract. In this paper, we studied maximal monotonicity of type FPV for sum of two maximal monotone operators of type FPV and the obtained results improve and complete the corresponding results of this filed.

Keywords: Maximal monotone operator, Maximal monotone operator of type FPV, Subdifferential.

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1. Introduction and Preliminaries

Throughout this paper, we assume that X is a real Banach space with norm $\|.\|$, that X^* is the continuous dual of X, and that X and X^* are paired by $\langle ., . \rangle$. Let $A : X \rightrightarrows X^*$ be a set-valued operator(also known as multifunction) from X to X^*, i.e., for every $x \in X$, $Ax \subseteq X^*$, and let $\operatorname{Gph} A = \{(x, x^*) \in X \times X^* | x^* \in Ax\}$ be the graph of A.

Definition 1.1. A is monotone if

$$\langle x - y, x^* - y^* \rangle \geq 0 \ \forall (x, x^*) \in \operatorname{Gph} A \ \forall (y, y^*) \in \operatorname{Gph} A,$$

and maximal monotone if A is monotone and A has no proper monotone extension (in the sense of graph inclusion).

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Definition 1.2. Let $A : X \rightrightarrows X^*$ be maximal monotone. A is of type FPV if for every open convex set $U \subset X$ such that $U \cap \operatorname{Dom} A \neq \emptyset$, the implication

$$x \in U \text{ and } \quad (x, x^*) \quad | \text{ is monotonically related to } \operatorname{Gph} A \cap U \times X^* \Rightarrow (x, x^*) \in \operatorname{Gph} A$$

holds.

Monotone operators have proven to be a key class of objects in modern Optimization and Analysis; see, e.g., the books [1, 2, 3, 4, 8, 12] and the references there in. We adopt standard notation used in these books: $\operatorname{Dom} A = \{x \in X \mid Ax \neq \emptyset\}$ is the domain of A. Given a subset C of X, $\operatorname{int} C$ is the interior of C, and $\overline{C}$ is the norm closure of C. The indicator function of C, written as ι_C, is defined at $x \in X$ by

$$\iota_C(x) = \begin{cases} 0 & x \in C \\ \infty & x \notin C. \end{cases}$$

We set $dist(x, C) = inf_{c \in C} \|x - c\|$, for $x \in X$. If $D \subseteq X$, we set $C - D = \{x - y \mid x \in C, y \in D\}$. For every $x \in X$, the normal cone operator of C at x is defined by $N_C(x) = \{x^* \in X^* \mid \sup_{c \in C} \langle c - x, x^* \rangle \leq 0\}$, if $x \in C$; and $N_C(x) = \emptyset$, if $x \notin C$. For $x, y \in X$, we set $[x, y] = \{tx + (1 - t)y \mid 0 \leq t \leq 1\}$. Given $f : X \to]-\infty, +\infty]$, we set $\operatorname{dom} f = f^{-1}(\mathbb{R})$ and $f^* : X^* \to [-\infty, +\infty] : x^* \mapsto \sup_{x \in X}(\langle x, x^* \rangle - f(x))$ is the Fenchel conjugate of f. if f is convex and $\operatorname{dom} f \neq \emptyset$, then $\partial f : X \rightrightarrows X^* : x \mapsto \{x^* \in X^* \mid (\ \forall y \in X)\langle y - x, x^* \rangle + f(x) \leq f(y)\}$ is the subdifferential operator of f. Finally, the open unit ball in X is denoted by $B_X = \{x \in X \mid \|x\| < 1\}$.
Let A and B be maximal monotone operators from X to X^*. Clearly, the sum operator $A + B : X \rightrightarrows X^* : x \mapsto Ax + Bx = \{a^* + b^* \mid a^* \in Ax, b^* \in Bx\}$ is monotone. Rockafellars guarantees maximal monotonicity of $A+B$ under Rockafellars constraint qualification $\operatorname{Dom} A \cap int \operatorname{Dom} B \neq \emptyset$ when X is reflexive- this result is often referred to as the sum theorem.

Theorem 1.3. [12] *Let $f : X \to]-\infty, +\infty]$ be a convex and lower semicontinuous function. Then f is continuous at the points of* $\operatorname{int} \operatorname{dom} f$.

Theorem 1.4. [7] *Let $f : X \to]-\infty, +\infty]$ be a proper, convex and lower semicontinuous function. Then ∂f is maximal monotone.*

Theorem 1.5. *(Rockafellar)*[6, 8, 12] *Let $f, g : X \to]-\infty, +\infty]$ be proper convex functions. Assume that there exists a point $x_0 \in \operatorname{dom} f \cap \operatorname{dom} g$ such that g is continuous at x_0. Then $\partial(f + g) = \partial f + \partial g$.*

Theorem 1.6. *(Verona-Verona)*[8, 10] *Let $f : X \to]-\infty, +\infty]$ be proper, lower semicotinuous, and convex, and let $B : X \rightrightarrows X^*$ be maximal monotone with full domain. Then $\partial f + B$ is maximal monotone.*

Theorem 1.7. *(Heisler)*[5] *Let $A, B : X \rightrightarrows X^*$ be maximal monotone with full domain. Then $A + B$ is maximal monotone.*

Now we cite some results on maximal monotone operators of type FPV

Theorem 1.8. *(Simons and Verona- Verona)*[8, 9] *Let $A : X \rightrightarrows X^*$ be a maximal monotone. Suppose that for every closed subset C of X with $\operatorname{Dom} A \cap \operatorname{int} C \neq \emptyset$, the operator $A + N_C$ is maximal monotone. Then A is of type FPV.*

Corollary 1.9. [11] *Let $A : X \rightrightarrows X^*$ be maximal monotone of type FPV with convex domain, let C be a nonempty closed convex subset of X, and suppose that $\operatorname{Dom} A \cap \operatorname{int} C \neq \emptyset$. Then $A + N_C$ is of type FPV.*

Theorem 1.10. [11] *Let $A, B : X \rightrightarrows X^*$ be maximal monotone with $\operatorname{Dom} A \cap \operatorname{int} \operatorname{Dom} B \neq \emptyset$. Assume that $A + N_{\overline{\operatorname{Dom} B}}$ is maximal monotone of type FPV, and $\operatorname{Dom} A \cap \overline{\operatorname{Dom} B} \subseteq \operatorname{Dom} B$. Then $A + B$ is maximal monotone.*

Theorem 1.11. [11] *Let $A : X \rightrightarrows X^*$ be maximal monotone of type FPV, and let $B : X \rightrightarrows X^*$ be maximal monotone with full domain. Then $A + B$ is maximal monotone.*

Theorem 1.12. [11] *Let $A : X \rightrightarrows X^*$ be maximal monotone of type FPV with convex domain, and let $B : X \rightrightarrows X^*$ be maximal monotone with $\operatorname{Dom} A \cap \operatorname{int} \operatorname{Dom} B \neq \emptyset$. Assume that $\operatorname{Dom} A \cap \overline{\operatorname{Dom} B} \subseteq \operatorname{Dom} B$. Then $A + B$ is maximal monotone .*

2. Main results

Theorem 2.1. *Let $A : X \rightrightarrows X^*$ be maximal monotone of type FPV with convex domain, and let $B : X \rightrightarrows X^*$ be maximal monotone with $\operatorname{Dom} A \cap \operatorname{int} \operatorname{Dom} B \neq \emptyset$. Assume that $\operatorname{Dom} A \cap \overline{\operatorname{Dom} B} \subseteq \operatorname{Dom} B$. Then $A + B$ is maximal monotone of type FPV.*

Proof. By Theorem 1.12 $A+B$ is maximal monotone and we it is proved that $A + B$ is of type FPV. Let D be a nonempty closed convex subset of X, and suppose that $\operatorname{Dom}(A + B) \cap \operatorname{int} D \neq \emptyset$. Let $x_1 \in \operatorname{Dom} A \cap \operatorname{int} \operatorname{Dom} B$ and $x_2 \in \operatorname{Dom}(A + B) \cap \operatorname{int} D$. Thus, there exists $\delta > 0$ such that $x_1 + \delta B_X \subset \operatorname{Dom} B$ and $x_2 + \delta B_X \subset D$. Then for small enough $\lambda \in]0, 1[$, we have $x_2 + \lambda(x_1 - x_2) + \frac{1}{2}\delta B_X \subset D$. Clearly, $x_2 + \lambda(x_1 - x_2) + \lambda\delta B_X \subset \operatorname{Dom} B$. Thus $x_2 + \lambda(x_1 - x_2) + \frac{\lambda\delta}{2} B_X \subset \operatorname{Dom} B \cap D$. Since $\operatorname{Dom} A$ is convex, $x_2 + \lambda(x_1 - x_2) \in \operatorname{Dom} A$ and $x_2 + \lambda(x_1 - x_2) \in \operatorname{Dom} A \cap \operatorname{int}(\operatorname{Dom} B \cap D)$. Then $\operatorname{Dom} A \cap \operatorname{int}(\operatorname{Dom} B \cap D) \neq \emptyset$ and by Theorem 1.9, $A + N_{\overline{\operatorname{Dom} B \cap D}}$ is maximal monotone of type FPV. By Theorem 1.5, $A + N_{\overline{\operatorname{Dom} B \cap D}} = (A + N_D) + N_{\overline{\operatorname{Dom} B}}$ is maximal monotone of type FPV. Now, by Theorem 1.10, $(A+N_D)+B$ is maximal monotone and hence, by Theorem 1.8, $A+B$ is of type FPV. □

Corollary 2.2. *Let $A, B : X \rightrightarrows X^*$ be maximal monotone with full domain. Then $A+B$ is maximal monotone of type FPV.*

Proof. By Theorem 1.7 $A+B$ is maximal monotone and clearly, all conditions of Theorem 2.1 are satisfied. Then $A+B$ is maximal monotone of type FPV. □

Corollary 2.3. *Let $A : X \rightrightarrows X^*$ be maximal monotone of type FPV with convex domain, let $f : X \to]-\infty, +\infty]$ be proper, convex and lower semicontinuous with $\operatorname{Dom} A \cap \operatorname{intDom} \partial f \neq \emptyset$. Assume that $\operatorname{Dom} A \cap \overline{\operatorname{Dom} \partial f} \subseteq \operatorname{Dom} \partial f$. Then $A + \partial f$ is maximal monotone of type FPV.*

Proof. By Theorem 1.4, ∂f is maximal monotone. The conclusion follows from assumptions and Theorem 2.1 □

Corollary 2.4. [11] *Let $A : X \rightrightarrows X^*$ be maximal monotone of type FPV with convex domain, let C be a nonempty closed convex subset of X, and suppose that $\operatorname{Dom} A \cap \operatorname{int} C \neq \emptyset$. Then $A + N_C$ is of type FPV.*

Proof. Let $f = \iota_C$, then all conditions of Corollary 2.3 are satisfied and hence $A + N_C$ is of type FPV. □

Theorem 2.5. *Let $A : X \rightrightarrows X^*$ be maximal monotone of type FPV with convex domain, and let $B : X \rightrightarrows X^*$ be maximal monotone with full domain. Then $A+B$ is maximal monotone of type FPV.*

Proof. By corollary 1.11, $A+B$ is maximal monotone. Let D be a nonempty closed convex subset of X, and suppose that $\operatorname{Dom}(A+B) \cap \operatorname{int} D \neq \emptyset$. By Theorem 1.9, $(A+N_D)+N_{\overline{\operatorname{Dom} B}} = A+N_D+N_X = A+N_D$ is maximal monotone of type FPV. Then Theorem 1.10 implies that $(A+N_D)+B = (A+B)+N_D$ is maximal monotone. Now, by Theorem 1.8, $A+B$ is of type FPV. □

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