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Involution Matrices of Real Quaternions

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Abstract. An involution or anti-involution is a self-inverse linear mapping. In this paper, we will present two real quaternion matrices, one corresponding to a real quaternion involution and one corresponding to a real quaternion anti-involution. Moreover, properties and geometrical meanings of these matrices will be given as reflections in $\mathbb{R}^3$.

Keywords: Real quaternions, Involutions, Anti-involutions.

2010 Mathematics subject classification: 11R52, 15A04, 15A33.

1. Introduction

Quaternions are an extension of the complex numbers $\mathbb{C}$ and were first described by Irish mathematician Sir William Rowan Hamilton in 1843. Hamilton was looking for a way to formalize 3 points in 3-space in the same way that 2 points can be defined in the complex field $\mathbb{C}$. If he had been able to find a way to formalize 3 points in 3-space, he would have effectively built a degree three field extension of real numbers $\mathbb{R}$ whose vector space forms the basis $\{1, i, j\}$ over $\mathbb{R}$ such that $i^2 = j^2 = -1$. For many years, he knew how to add and subtract 3 points in 3-space. However, he had been stuck on the problem of multiplication and division for over 10 years. Finally, the great breakthrough in quaternions

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came on 16 October 1843 in Dublin, while Hamilton was on a walk with his wife. They had been walking along the towpath of the Royal Canal when it occurred to Hamilton that the algebra of quaternions would require three rather than two imaginary parts satisfying the conditions

$$i^2 = j^2 = k^2 = ijk = -1.$$

Hamilton carved these results into the stone of Brougham Bridge.

Quaternions are widely used in computer graphic technology, physics, mechanics, electronics, kinematics, etc., since they are useful to perceive rotations, reflections and rigid body (screw) motions. The Book of Involutions by Knus, Merkurjev, Rost and Tignol is one of the most important sources about involutions. In this book first and second kind involutions are widely studied as algebraically for central simple algebras. Ell and Sangwine have studied involutions and anti-involutions of real quaternions with their geometrical meanings in $\mathbb{R}^3$ beside their algebraic meanings, see[1].

In this paper we will begin by reviewing basics of real quaternions and their matrix representations. After, definitions of the concepts involution and anti-involution will be given. Finally, we will give a matrix corresponding to a real quaternion involution, and a matrix corresponding to a real quaternion anti-involution, with their properties and geometrical meanings as reflections in $\mathbb{R}^3$.

2. PRELIMINARIES

In this section we will present basics of real quaternions and their matrix representations.

2.1. **Basics of Real Quaternions.** The real quaternion algebra

$$\mathbb{H} = \{q = w + xi + yj + zk : w, x, y, z \in \mathbb{R}\}$$

is a four dimensional vector space over the field of real numbers $\mathbb{R}$, with a basis $\{1, i, j, k\}$. Multiplication is defined by the hypercomplex operator rules

$$i^2 = j^2 = k^2 = ijk = -1.$$

It can be easily checked that those rules imply

$$ij = -ji = k, \; jk = -kj = i, \; ki = -ik = j.$$

Also, $\mathbb{H}$ is an associative and non-commutative division ring.

For any real quaternion $q = w + xi + yj + zk$, we define the scalar part and vector part as $S(q) = w$ and $V(q) = xi + yj + zk$, respectively.

The (quaternionic) conjugate of $q = S(q) + V(q)$ is $q = S(q) - V(q)$. If $S(q) = 0$ then q is called pure. Pure real quaternions set

$$\widehat{\mathbb{H}} = \{q = xi + yj + zk : w, x, y, z \in \mathbb{R}\}$$

is a linear subspace of $\mathbb{H}$ spanned by $\{i,\ j,\ k\}$.

For real quaternions $q = w_1 + x_1 i + y_1 j + z_1 k$ and $p = w_2 + x_2 i + y_2 j + z_2 k$, we define the sum of q and p ;multiplication of q with a real scalar λ; and the product of q and p as

$$q + p = (w_1 + w_2) + (x_1 + x_2)\, i + (y_1 + y_2)\, j + (z_1 + z_2)\, k$$

$$\lambda q = \lambda w_1 + \lambda x_1 i + \lambda y_1 j + \lambda z_1 k$$

$$qp = S\,(q)\, S\,(p) - \langle V(q), V(p)\rangle + S(q)V(p) + S(p)V(q) + V(q) \wedge V(p)$$

where $\langle V(q), V(p)\rangle = x_1x_2 + y_1y_2 + z_1z_2$, $V(q) \wedge V(p) = (y_1z_2 - z_1y_2)i - (x_1z_2 - z_1x_2)j + (x_1y_2 - y_1x_2)k$. It can be easily shown that $\overline{pq} = \overline{qp}$ and $\overline{p+q} = \overline{p} + \overline{q} = \overline{q} + \overline{p}$ for general real quaternions q and p, see [2, 3, 4].

The norm and modulus of a real quaternion $q = w_1 + x_1 i + y_1 j + z_1 k$ can be given, respectively, by $\|q\| = qq = qq = w_1^2 + x_1^2 + y_1^2 + z_1^2 \geq 0$ and $|q| = \sqrt{\|\ q\ \|}$. If $\|q\| = 1$ then q is said to be unit. The multiplicative inverse of any nonzero real quaternion is $q^{-1} = \dfrac{q}{\|q\|}$.

Hamilton showed that any unit pure real quaternion is a square root of -1. In other words, if μ is a unit pure real quaternion then $\mu^2 = -1$, see [5].

A real quaternion $q = w + xi + yj + zk$ can be given in complex form as $q = a + \mu b$ where $a = w$,$b = \sqrt{x^2 + y^2 + z^2}$ and $\mu = \dfrac{xi + yj + zk}{b}$.

2.2. **Matrix Representation of Real Quaternions.** Real quaternions can be represented in the form of 4×4 real matrices so that the matrix multiplication corresponds to real quaternion multiplication, see [6]. By using the hypercomplex operator rules, the coefficients of the real quaternion multiplication $r = qp = w_0 + x_0 i + y_0 j + z_0 k$ of two arbitrary real quaternions $q = w_1 + x_1 i + y_1 j + z_1 k$ and $p = w_2 + x_2 i + y_2 j + z_2 k$ can be given by

$$w_0 = w_1w_2 - x_1x_2 - y_1y_2 - z_1z_2$$

$$x_0 = x_1w_2 + w_1x_2 - z_1y_2 + y_1z_2$$

$$y_0 = y_1w_2 + w_1y_2 + z_1x_2 - x_1z_2$$

$$z_0 = z_1w_2 + w_1z_2 - y_1x_2 + x_1y_2.$$

Thus, the following two matrix-vector multiplication can be given:

(i) The standard form preserving the order of the multiplication as

$$\begin{bmatrix} w_0 \\ x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} w_1 & -x_1 & -y_1 & -z_1 \\ x_1 & w_1 & -z_1 & y_1 \\ y_1 & z_1 & w_1 & -x_1 \\ z_1 & -y_1 & x_1 & w_1 \end{bmatrix} \begin{bmatrix} w_2 \\ x_2 \\ y_2 \\ z_2 \end{bmatrix}.$$

In this form q is taken as a left-handed operator of variable p, where the operator q and variable p correspond, respectively, to the matrices

$$\begin{bmatrix} w_1 & -x_1 & -y_1 & -z_1 \\ x_1 & w_1 & -z_1 & y_1 \\ y_1 & z_1 & w_1 & -x_1 \\ z_1 & -y_1 & x_1 & w_1 \end{bmatrix} \text{ and } \begin{bmatrix} w_2 \\ x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

(ii) The transposed form reversing the order of the multiplication as

$$\begin{bmatrix} w_0 \\ x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} w_2 & -x_2 & -y_2 & -z_2 \\ x_2 & w_2 & z_2 & -y_2 \\ y_2 & -z_2 & w_2 & x_2 \\ z_2 & y_2 & -x_2 & w_2 \end{bmatrix} \begin{bmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{bmatrix}.$$

In this form p is taken as a right-handed operator of variable q, where the operator q and variable p correspond, respectively, to the matrices

$$\begin{bmatrix} w_2 & -x_2 & -y_2 & -z_2 \\ x_2 & w_2 & z_2 & -y_2 \\ y_2 & -z_2 & w_2 & x_2 \\ z_2 & y_2 & -x_2 & w_2 \end{bmatrix} \text{ and } \begin{bmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{bmatrix}.$$

3. INVOLUTION MATRICES OF REAL QUATERNIONS

In this section, fristly we will give the definitions of the concepts involution mapping and anti-involution mapping. After, we will present two matrices, one corresponding to a real quaternion involution and one corresponding to a real quaternion anti-involution, with their properties and geometrical meanings as reflections in $\mathbb{R}^3$.

Definition 3.1. A transformation f is an involution (also known as an involutory anti-automorphism) if it satisfy the following axioms:

Axiom 1. An involution is its own inverse (self-inverse) : $f(f(x)) = x$

Axiom 2. An involution is linear : $f(x_1 + x_2) = f(x_1) + f(x_2)$ and $\lambda f(x) = f(\lambda x)$, where λ is real constant.

Axiom 3. An involution is anti-automorphism : $f(x_1 x_2) = f(x_2) f(x_1)$.
To be an anti-involution (also known as an involutory automorphism) a transformation f must satisfy self-inverse linearity similiar to involution,

except that it does not obey Axiom 3 as stated. Instead, it must satisfy the following axiom:

Axiom 4. An anti-involution is homomorphic : $f(x_1x_2) = f(x_1)f(x_2)$, see [7].

3.1. **Involution Matrices of Real Quaternions.**

Proposition 3.2. *The transformation*

$$f_v : \mathbb{H} \to \mathbb{H}$$

defined by

$$q \to f_v(q) = -v\bar{q}v;\ v^2 = -1;\ v \in \widehat{\mathbb{H}}$$

where q is an arbitrary real quaternion and v is any unit pure real quaternion, is an involution, see [1].

The geometric interpretation in $\mathbb{R}^3$ of the involution $f_v(q) = -v\bar{q}v$ can be given by the following theorem.

Theorem 3.3. *For an arbitrary real quaternion $q = a + \mu b$, where $a, b \in \mathbb{R}$ and μ is a unit pure real quaternion, the involution $f_v(q) = -v\bar{q}v$, where v is any unit pure real quaternion, leaves the scalar part of q (that is, a) invariant, and reflects the vector part of q (that is, μb) in the plane normal to the axis of involution v, see [1]*

In $\mathbb{R}^3$,$v\mu v$ represent a reflection of μ in the plane normal to v, see[8]. The geometry of $v\mu v$ in $\mathbb{R}^3$ can be given by Figure 1:

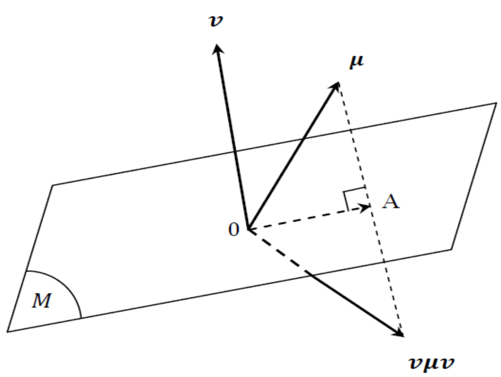

Figure 1.

In Figure 1, M is the plane which is perpendicular to v, $\vec{A}$ is the orthogonal projection vector of μ on M and $|v| = |\mu| = |v\mu v| = 1$.

Now, we will obtain the matrix corresponding to the involution $f_v(q) = -v\bar{q}v$.

Let $q = a+\mu b$ be an arbitrary real quaternion and $v = xi+yj+zk$ any unit pure real quaternion. Using the involution transformation $f_v(q) =$

$-v\bar{q}v$ and the basis $\{1, i, j, k\}$ of the vector space $\mathbb{H}$, we can write the equations

$f_v(1) = -v\bar{1}v = -v^2 = 1,$
$f_v(i) = -v\bar{i}v = (1-2x^2)i - 2xyj - 2xzk,$
$f_v(j) = -v\bar{j}v = -2xyi + (1-2y^2)j - 2yzk,$
$f_v(k) = -v\bar{k}v = -2xzi - 2yzj + (1-2z^2)k$ Hence, the matrix representation corresponding to involution $f_v(q) = -v\bar{q}v$ can be given as

$$Mq = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-2x^2 & -2xy & -2xz \\ 0 & -2xy & 1-2y^2 & -2yz \\ 0 & -2xz & -2yz & 1-2z^2 \end{bmatrix} \begin{bmatrix} a \\ \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$$

where $q = a + \mu b$ corresponds to the 4×1 matrix, M corresponds to the 4×4 matrix and μb is equal to (μ_1, μ_2, μ_3). It can be easily checked that M is orthogonal (i.e. $MM^T = I$) and symmetric (i.e. $M = M^T$) with determinant -1, so that the involution $f_v(q) = -v\bar{q}v$ represents a reflection in $\mathbb{R}^4$.

The product Mq, leaves the scalar part of q (that is, a) invariant and in $\mathbb{R}^3$ reflects the vector part of q (that is, μb) in the plane normal to the axis of involution v.

Example 3.4. For real quaternion $q = 3 + 2(\frac{3}{5}, 0, \frac{4}{5})$ and unit pure real quaternion $v = (0, 0, 1)$ the matrix representation corresponding to involution $f_v(q) = -v\bar{q}v$ can be given with

$$Mq = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ \frac{6}{5} \\ 0 \\ \frac{8}{5} \end{bmatrix} = (3, \frac{6}{5}, 0, -\frac{8}{5}).$$

The effect of the matrix M on the real quaternion q is: It leaves the scalar part of q (that is, 3) invariant and reflects the vector part of q (that is, $2(\frac{3}{5}, 0, \frac{4}{5})$) in the plane normal to the axis of involution $v = (0, 0, 1)$ in $\mathbb{R}^3$, see Figure 2.

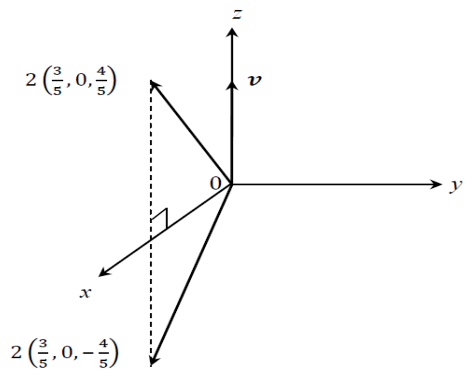

Figure 2.

Corollary 3.5. *For pure real quaternion* $q = \mu_1 i + \mu_2 j + \mu_3 k = (\mu_1, \mu_2, \mu_3)$ *and unit pure real quaternion* $v = xi + yj + zk = (x, y, z)$ *the matrix product*

$$\begin{bmatrix} 1-2x^2 & -2xy & -2xz \\ -2xy & 1-2y^2 & -2yz \\ -2xz & -2yz & 1-2z^2 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$$

reflects q *in the plane normal to* v *in* $\mathbb{R}^3$.

3.2. Anti-Involution Matrices of Real Quaternions.

Proposition 3.6. *The transformation*

$$f_v : \mathbb{H} \to \mathbb{H}$$

defined by

$$q \to f_v(q) = -vqv; \;\; v^2 = -1; \;\; v \in \widehat{\mathbb{H}}$$

where q is an arbitrary real quaternion and v is any unit pure real quaternion, is an anti-involution, see [1].

The geometric interpretation in $\mathbb{R}^3$ of the anti-involution $f_v(q) = -vqv$ can be given by the following theorem.

Theorem 3.7. *For an arbitrary real quaternion* $q = a + \mu b$*, where* $a, b \in \mathbb{R}$ *and* μ *is a unit pure real quaternion, the anti-involution* $f_v(q) = -vqv$*, where* v *is any unit pure real quaternion, leaves the scalar part of* q *(that is,* a*) invariant, and reflects the vector part of* q *(that is,* μb*) in the line defined by the axis of involution (equivalently to rotate the vector part by* π *about the axis of involution).* $f_v(q) = -v\bar{q}v$ *is the conjugate of* $f_v(q) = -vqv$*, see [1].*

The geometry of $-v\mu v$ in $\mathbb{R}^3$ can be given by Figure 3:

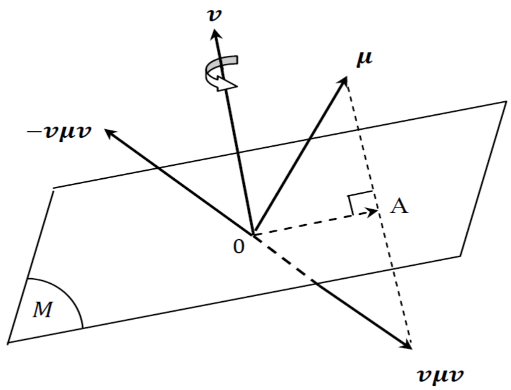

Figure 3.

In Figure 3, M is the plane which is perpendicular to v, $\vec{A}$ is the orthogonal projection vector of μ on M and $|v| = |\mu| = |v\mu v| = |-v\mu v| = 1$.

Now, we will obtain the matrix corresponding to the anti-involution $f_v(q) = -vqv$. Let $q = a + \mu b$ be an arbitrary real quaternion and $v = xi+yj+zk$ any unit pure real quaternion. Using the anti-involution transformation $f_v(q) = -vqv$ and the basis $\{1, i, j, k\}$ of the vector space $\mathbb{H}$, we can write the equations

$f_v(1) = -v1v = -v^2 = 1,$

$f_v(i) = -viv = -(1-2x^2)i + 2xyj + 2xzk,$

$f_v(j) = -vjv = 2xyi - (1-2y^2)j + 2yzk,$

$f_v(k) = -vkv = 2xzi + 2yzj - (1-2z^2)k.$

Hence, the matrix representation corresponding to anti-involution $f_v(q) = -vqv$ can be given as

$$Mq = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2x^2-1 & 2xy & 2xz \\ 0 & 2xy & 2y^2-1 & 2yz \\ 0 & 2xz & 2yz & 2z^2-1 \end{bmatrix} \begin{bmatrix} a \\ \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$$

where $q = a + \mu b$ corresponds to the 4×1 matrix, M corresponds to the 4×4 matrix and μb is equal to (μ_1, μ_2, μ_3). It can be easily checked that M is orthogonal (i.e. $MM^T = I$) and symmetric (i.e. $M = M^T$) with determinant $+1$, so that the anti-involution $f_v(q) = -vqv$ represents a rotation in $\mathbb{R}^4$.

The product Mq, leaves the scalar part of q (that is, a) invariant and in $\mathbb{R}^3$ reflects the vector part of q (that is, μb) in the line defined by the axis of involution v (equivalently to rotate the vector part by π about the axis of involution).

Example 3.8. For real quaternion $q = -1 + 7(\frac{1}{\sqrt{3}}, -\frac{1}{2}, \frac{\sqrt{5}}{2\sqrt{3}})$ and unit pure real quaternion $v = (0, 0, 1)$ the matrix representation corresponding to anti-involution $f_v(q) = -vqv$ can be given as

$$Mq = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ \frac{7}{\sqrt{3}} \\ -\frac{7}{2} \\ \frac{7\sqrt{5}}{2\sqrt{3}} \end{bmatrix} = (-1, -\frac{7}{\sqrt{3}}, \frac{7}{2}, \frac{7\sqrt{5}}{2\sqrt{3}}).$$

The effect of the matrix M on the real quaternion$q$ is: It leaves the scalar part of q (that is, -1) invariant and in $\mathbb{R}^3$ reflects the vector part of q (that is, $7(\frac{1}{\sqrt{3}}, -\frac{1}{2}, \frac{\sqrt{5}}{2\sqrt{3}})$) in the line defined by the axis of involution v (equivalently to rotate the vector part by π about the axis of involution) in $\mathbb{R}^3$, see Figure 4.

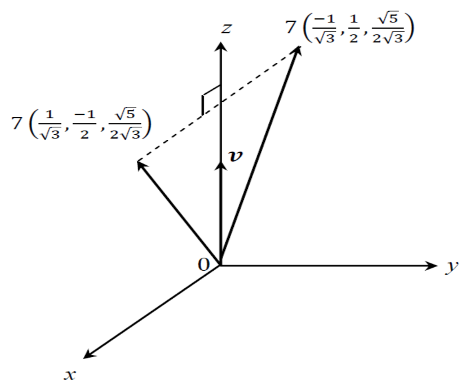

Figure 4.

Corollary 3.9. *For pure real quaternion* $q = \mu_1 i + \mu_2 j + \mu_3 k = (\mu_1, \mu_2, \mu_3)$ *and unit pure real quaternion* $v = xi + yj + zk = (x, y, z)$ *the matrix product*

$$\begin{bmatrix} 2x^2 - 1 & 2xy & 2xz \\ 2xy & 2y^2 - 1 & 2yz \\ 2xz & 2yz & 2z^2 - 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$$

reflects q *in the line* v *(equivalently to rotate* q *by* π *about the axis* v*) in* $\mathbb{R}^3$.

4. CONCLUSION

The matrix representation corresponding to involution$f_v(q) = -v\bar{q}v$ represents a reflection in $\mathbb{R}^4$, and in $\mathbb{R}^3$ it reflects the vector part of q in the plane normal to the axis of involution v. Also, the matrix representation corresponding to ant-involution $f_v(q) = -vqv$ represents a rotation in $\mathbb{R}^4$, and in $\mathbb{R}^3$ it reflects the vector part of q in the line defined by the axis of involution (equivalently to rotate the vector part by π about the axis of involution).

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