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Permanency and Asymptotic Behavior of The Generalized Lotka-Volterra Food Chain System

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Abstract.
In the present paper a generalized Lotka-Volterra food chain system has been studied and also its dynamic behavior such as locally asymptotic stability has been analyzed in case of existing inter-species competition. Furthermore, it has been shown that the said system is permanent (in the sense of boundedness and uniformly persistent). Finally, it is proved that the nontrivial equilibrium point of the above system is locally asymptotically stable.

Keywords: Permanence, Lotka-Volterra Model, Stability, Boundedness.

2010 Mathematics subject classification: 34C25, 34C30, 34D45.

1. Introduction

In the 1920's, A.J. Lotka proposed predator-prey equations that were the mainstay of community and ecosystem modeling, but shortly afterwards, V. Volterra (1928) independently discovered the same predator-prey model, which thereafter became known as the Lotka-Volterra equations. The Lotka-Volterra model consists of the following system of differential equations:

$$\begin{cases} x' = x(a - by) \\ y' = y(-c + dx) \end{cases}$$

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where $y(t)$ and $x(t)$ represent the predator population and the prey population as functions of time, respectively, and all parameters a,b,c,d are positive [1,2,3]. The three species food chain model is an extension of the general two species model given by Lotka and Volterra [3]. We consider the nonlinear three species food chain model consisting of a prey, predator and super predator:

$$\begin{cases} x_1' = x_1(r_1 - a_{11}x_1 - a_{12}x_2) \\ x_2' = x_2(-r_2 + a_{21}x_1 - a_{22}x_2 - a_{23}x_3) \\ x_3' = x_3(-r_3 + a_{32}x_2 - a_{33}x_3) \end{cases} \tag{1.1}$$

all parameters r_j and a_{ij} are real positive constants.

Having considered $i, j = 1, 2, 3$; the parameter r_i represents the natural growth rate of the i^{th} population, and coefficients a_{ij} describes the effect of j^{th} upon the i^{th} population, which is positive provided it enhances and negative if it inhibits the growth [4].

For convenience, the lowest-level prey was denoted by $x(t)$ that is preyed by a mid-level species $y(t)$, which is preyed by a top level predator $z(t)$ where $\{(x, y, z) \in R^3 : x, y, z \in R^+\}$.

Therefore, system (1.1) was denoted by the following system:

$$\begin{cases} x' = x(a - bx - cy) \\ y' = y(-d + ex - fy - gz) \\ z' = z(-h + iy - jz) \end{cases} \tag{1.2}$$

In this paper, we consider generalization of three species food chain as following

$$\begin{cases} x' = x(t)(f(x(t)) - p(y(t))), \\ y' = y(t)(-g(y(t)) + q(x(t)) - h(z(t))), \\ z' = z(t)(-k(z(t)) + i(y(t))). \end{cases} \tag{1.3}$$

where $x(t)$, $y(t)$ and $z(t)$ respectively denote the densities of prey, predator and top predator. The functions f, g, p, q, h, k, i are continuous and differentiable.

The function $f(x)$ is the specific growth rate of the prey species in the absence of predator species. Also, there exists competition between preys to earn food, we assume $\dfrac{df}{dx} < 0$.

The function $p(y)$ is efficiency of the predator species on the prey species in a time unit. Since the predators have a negative efficiency on the prey, therefore we put a negative sign on the back of the function $p(y)$ and $\frac{dp}{dy} > 0$.

The function $g(y)$ is the specific growth rate of the predator in the absence of prey species. Since the predator dies in the absence of prey, we put a negative sign on the back of the function $g(y)$ and $\frac{dg}{dy} > 0$.

Definition 1.1. The system (1.1) is said to be permanent (bounded and uniformly persistent) if there exist positive constants $m_1, m_2.m_3$ and M_1, M_2, M_3 such that every positive solution $(x_1(t), x_2(t), x_3(t))^T$ of (1.1) satisfies $m_1 \leq liminf_{t\longrightarrow\infty} x_1(t) \leq limsup_{t\longrightarrow\infty} x_1(t) \leq M_1$, and $m_2 \leq liminf_{t\longrightarrow\infty} x_2(t) \leq limsup_{t\longrightarrow\infty} x_2(t) \leq M_2$ and $m_3 \leq liminf_{t\longrightarrow\infty} x_3(t) \leq limsup_{t\longrightarrow\infty} x_3(t) \leq M_3$.

We show the efficiency of the prey species on predator species with the function $q(x)$ as positive since prey species has positive efficiency on predator species, therefore, the sign of function $q(x)$ is positive.

For the top predator, the function k, h define similarity.

2. Main results

In this section, we show that system 1.1 is permanence and we will analyze the stability of this system.

The system (1.1) has the nontrivial equilibrium $E = (\overline{x}, \overline{y}, \overline{z})$. We consider the linearized system of (1.4) at $(\overline{x}, \overline{y}, \overline{z})$. The Jacobian matrix at the $(\overline{x}, \overline{y}, \overline{z})$ is

$$J = \begin{bmatrix} a\overline{x} & -b\overline{x} & 0 \\ c\overline{y} & -d\overline{y} & -e\overline{y} \\ 0 & m\overline{z} & -l\overline{z} \end{bmatrix}$$

where

$$\begin{cases} a = \dfrac{df}{dx} < 0, & d = \dfrac{dg}{dy} > 0, \quad m = \dfrac{di}{dy} > 0, \\ b = \dfrac{dp}{dy} > 0, & e = \dfrac{dh}{dz} > 0, \\ c = \dfrac{dq}{dx} > 0, & l = \dfrac{dk}{dz} > 0. \end{cases} \tag{2.1}$$

The characteristic equation of J is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \tag{2.2}$$

where

$$a_1 = l\overline{z} + d\overline{y} - a\overline{x},$$
$$a_2 = -ad\overline{xy} + ld\overline{yz} - al\overline{xz} + me\overline{yz} + bc\overline{xy}$$
$$a_3 = -adl\overline{xyz} - mea\overline{xyz}$$

Theorem 2.1. *Suppose that $E = (\overline{x}, \overline{y}, \overline{z})$ is the nontrivial equilibrium point of system (1.1). Then $(\overline{x}, \overline{y}, \overline{z})$ is locally asymptotically stable.*

Proof. According to Routh-Hurwitz criterion, $E = (\overline{x}, \overline{y}, \overline{z})$ is locally asymptotically stable provided the following conditions are satisfied:

$$a_2 > 0, a_1 > 0, a_3 > 0, a_1a_2 - a_3 > 0$$

Since l, d, b, c, m, e are positive and a is negative, we have

$$\begin{aligned}
a_1 &= l\overline{z} + d\overline{y} - a\overline{x} > 0, \\
a_3 &= -adl\overline{xyz} - mea\overline{xyz} > 0 \\
a_2 &= -ad\overline{xy} + ld\overline{yz} - al\overline{xz} + me\overline{yz} + bc\overline{xy} > 0
\end{aligned}$$

and

$$\begin{aligned}
a_1a_2 - a_3 = {} & a^2d\overline{x}^2\overline{y} - ad^2\overline{y}^2\overline{x} - 2adl\overline{xyz} + a^2l\overline{x}^2\overline{z} \\
& - al^2\overline{z}^2\overline{x} + d^2l\overline{y}^2\overline{z} + dl^2\overline{z}^2\overline{y} - abc\overline{x}^2\overline{y} \\
& + bcd\overline{y}^2\overline{x} + dem\overline{y}^2z + elm\overline{z}^2\overline{y} + bcl\overline{xyz} > 0
\end{aligned}$$

Therefore, the point $E = (\overline{x}, \overline{y}, \overline{z})$ is locally asymptotically stable. Therefore, the proof is done. □

Note: Let h be a function. We denote $h^u := maxh$ and $h^l := minh$.

Theorem 2.2. *The system (1.1) is permanent*

Proof. Let $(x_1(t), x_2(t), x_3(t))^T$ be as a positive solution of the system (1.1) with positive initial conditions $x_1(0), x_2(0), x_3(0) > 0$.

From the first equation of (1.1) and positivity of the solutions of it , we have $\frac{dx}{dt} \leq x(t)[f^u(x) - p^l(y)]$. Let $\varphi = \frac{1}{x}$. Thus $\frac{d\varphi}{dt} \geq \varphi[p^l(y) - f^u(x)]$. Therefore, $\varphi(t) \geq \varphi(0)exp[p^l(y) - f^u(x)]ds = \varphi(0)expAt$. Then $x(t) \leq \frac{1}{x(0)}expAt$. Regarding the system (1.1) the term $f(x) - p(y)$ is species growth rate of prey species and prey growth rate is not negative and on the other hand, it is carrying capacity for environment, thus $A = [p^l(y) - f^u(x)]$ is less than zero. Therefor there is $M_1 > 0$ such that $limsupx(t) \leq M_1$. So $x(t)$ is upper bounded for $t \geq 0$. Similarly from the second equation and third equation of (1.1), we can prove that $\limsup y(t) \leq M_2, \limsup z(t) \leq M_3$ that means $y(t), z(t)$ are upper bounded for $t \geq 0$.

Now we prove that there are m_1, m_2 and m_3 such that m_1 is the lower bound of x, m_2 is lower bound of y and m_3 is lower bound of z. By using the first equation of system (1.1), we have: $\frac{dx}{dt} \geq x[f^l(x) - p^u(y)]$ Thus $x(t) \geq x(0)exp[f^l(x) - p^u(y)]ds = x(0)exp[f^l(x) - p^u(y)]t = x(0)expBt$. On the other hand $limsupx(t) \leq M_1$, then there is $m_1 > 0$ such that $\liminf x(t) \geq m_1$. Similarly, we can prove that there are $m_2, m_3 > 0$ such that $liminfy(t) \geq m_2, liminfz(t) \geq m_3$. Hence the above argument implies that the solutions of the system (1.1) are bounded and uniformly the persistent, which proves the permanence. □

3. Conclusion

Three (or higher)-dimensional models in general are difficult to analyze as far as the detailed behavior of their solutions is concerned. Consequently, the notion of the persistence has been used to describe the situation when all interacting populations in a given system survive. Boundedness of a model guarantees its validity and it is necessary to determine persistence of the model.

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