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Dynamics of a Discrete-Time Plant-Herbivore Model

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Abstract. In this paper, we examine a discrete-time plant-herbivore model. We investigate stability of model

$$\begin{cases} x_{t+1} = x_t e^{r[1-x_t]-ay_t}, \\ y_{t+1} = x_t e^{r[1-x_t]}[1-e^{-ay_t}]. \end{cases}$$

Phase portraits are drawn for different ranges of parameters. We use the Liapunov-Schmidt reduction for attain a simpler and smaller system. Transition route to chaos dynamics is established via period-doubling bifurcations. Conditions of occurrence the period-doubling, Neimark-Sacker and saddle-node bifurcations are analysed. We study stable and unstable manifolds for this system in equilibrium points. Without the herbivore, the plant population follows the dynamics of the Ricker model.

Keywords: Stability, Liapunov-Schmidt reduction , Manifold, Bifurcation.

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1. Introduction

Plant-herbivore interactions exhibit natural oscillations in the populations of both Plant and the herbivore. These discrete-time models can

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describe occurring over discrete time steps. These models are a group of host-parasitoid systems. The simplest version of host-parasitoid systems is Nicholson-Bailey in 1935. In this paper, we examine a discrete-time plant-herbivore model by the following form

$$\begin{cases} x_{t+1} = x_t e^{r[1-x_t]-ay_t}, \\ y_{t+1} = x_t e^{r[1-x_t]}[1-e^{-ay_t}]. \end{cases} \tag{1.1}$$

Here, x_t is the population of plant and y_t is the population of herbivore. Parameter r is growth rate of plants and a measures average area of leaves consumed by a herbivore. We assume the herbivore does not attack the plant before the plant grows.
This model was analysed by Yun Kang, Dieter Armbruster and Yang Kuang in 2008 [1].
If the herbivore attacks the plant before the plant grows, we will have,

$$\begin{cases} x_{t+1} = x_t e^{r[1-x_t]-ay_t}, \\ y_{t+1} = x_t[1-e^{-ay_t}]. \end{cases}$$

We focus on the first case. We use the Liapunov-Schmidt reduction for attain a simpler and smaller system. This method replaces a large and complicated systems by a simpler ones which contain all the essential information concering a bifurcation [2]. This simple model exhibits both the period-doubling and the quasi-periodic routes to chaos. We study this model with a perturbation on plant dynamics. We add a constant immigration term as a proportion of the average density to the plant population in (1). Without the herbivore, the plant population follows the dynamics of the Ricker model. Various types of dynamics having stable fixed point, chaotic bands, periodic windows in this case [3].

2. Solutions and Local Asymptotic Stability

In this section, we study the equilibrium solutions and the local asymptotic stability of the model (1).
To analysis the local asymptotically stability, Jacobian matrix is calculated and evaluated at equilibrium.
In model (1), we have one extinction fixed point (0,0), one exclusion fixed point $(1,0)$ and one coexistence fixed point.
The fixed point $(0,0)$ represents the extinction of both species. The Jacobian matrix for system (1), has the following form:

$$\mathbf{J} = \begin{pmatrix} e^{r(1-x)-ay}(1-rx) & -axe^{r(1-x)-ay} \\ e^{r(1-x)}(1-e^{-ay})(1-rx) & axe^{r(1-x)-ay} \end{pmatrix}.$$

Jacobian matrix at $(0,0)$ is calculated:

$$\mathbf{J}\mid_{(\mathbf{0},\mathbf{0})} = \begin{pmatrix} e^r & 0 \\ 0 & 0 \end{pmatrix}.$$

The characteristic polynomial for $J(0,0)$ is the following form,

$$P_0(\lambda) = \lambda^2 - e^r\lambda.$$

Note. If λ_i are eigenvalues for Jacobian matrix at the equilibrium point and we have $|\lambda_i| < 1$, then the equilibrium is locally asymptotically stable.
The eigenvalues are $\lambda_{1,2} = e^r, 0$. Since $r > 0$, and hence $e^r > 1$, the eigenvalue e^r is greater than 1 that corresponds to a one-dimensional unstable manifold and $\lambda_2 = 0$ correspond to a one-dimensional stable manifold.

Theorem 2.1. *The extinction fixed point (0,0) of the system (1) is a saddle point and unstable.*

The Jacobian evaluated at the fixed point $(1,0)$ is given by:

$$\mathbf{J}\mid_{(\mathbf{1},\mathbf{0})} = \begin{pmatrix} 1-r & -a \\ 0 & a \end{pmatrix}.$$

The eigenvalues of $J(1,0)$ are $\lambda_1 = a$, $\lambda_2 = 1-r$.

Theorem 2.2. *(Jury condition)[4] The axial fixed point* $(1,0)$ *of system (1) is asymptotically stable if we have,*

$$|a| < 1, \quad r < 2 \quad \textit{and} \quad (1-r) < \frac{1}{a}.$$

Here, we applied the Liapunov-Schemidt reduction to reduce. This will be accomplished by means of a perturbaton expansion near a bifurcation point of (1). It can be shown that solutions of these reduced equations are locally in one to one correspondence to the solutions of (1) [2].
We make a change of variables. Let $x = u + 1$ and $y = v$. With this change of variables, $(1,0)$ coincide with origin and System (1) by the following form,

$$\begin{cases} u_{t+1} = -1 + (1+u_t)e^{-ru_t - av_t}, \\ v_{t+1} = (1+u_t)e^{-ru_t}(1 - e^{-av_t}). \end{cases}$$

We assume that $u_{t+1} = u_t$. Then, we have,

$$(u_t + 1) = (1+u_t)e^{-ru_t - av_t} \quad \Longrightarrow \quad e^{-ru_t - av_t} = 1 \quad \Longrightarrow \quad e^{-ru_t} = e^{av_t}.$$

and

$$-ru_t = av_t \quad \Longrightarrow \quad u_t = \frac{-av_t}{r}.$$

Consequently,

$$\begin{aligned}
v_{t+1} &= (1+u_t)e^{-ru_t}(1-e^{-av_t}) = (1+u_t)e^{av_t}(1-e^{-av_t}) = (1+u_t)(e^{av_t}-1), \\
v_{t+1} &= (1-\frac{av_t}{r})(e^{av_t}-1) = (e^{av_t}-1) - \frac{av_t}{r}(e^{av_t}-1), \\
v_{t+1} &= (av_t + \frac{a^2v_t^2}{2!} + \frac{a^3v_t^3}{3!} + O(4)) - \frac{av_t}{r}(av_t + \frac{a^2v_t^2}{2!} + \frac{a^3v_t^3}{3!} + O(4)),
\end{aligned}$$

Then, we have

$$v_{t+1} = av_t + a^2v_t^2(\frac{1}{2!} - \frac{1}{r}) + a^3v_t^3(\frac{1}{3!} - \frac{1}{2!}\frac{1}{r}) + O(4),$$

We assume,

$$G(v) = av + a^2v^2(\frac{1}{2!} - \frac{1}{r}) + a^3v^3(\frac{1}{3!} - \frac{1}{2!}\frac{1}{r}), \tag{2.1}$$

In equation (2), we have three fixed points, one fixed point $\bar{v}_1 = 0$ and,

$$\bar{v}_2 = \frac{-1}{2}\frac{3ar - 6a - \sqrt{-15a^2r^2 + 36a^2r + 36a^2 + 24ar^2 - 72ar}}{(r-3)a^2},$$

$$\bar{v}_3 = \frac{-1}{2}\frac{3ar - 6a + \sqrt{-15a^2r^2 + 36a^2r + 36a^2 + 24ar^2 - 72ar}}{(r-3)a^2}.$$

Notice that the equilibriums $\bar{v}_2 < 0$ is biologically uninteresting. We will later applied eq. (2) as the bifurcation equation. The stability of this system is affected by 3 factors, 1. the herbivore populations, 2. r, the growth rate of plant, 3. a, total amount of the biomass that an herbivore consumes.

The coexistence equilibrium (x^*, y^*) for system (1) is $x^* = \frac{r-ay^*}{r}$ and y^* is the positive root of the equation, $y^* = x^*(e^{ay^*} - 1)$.

The Jacobian at the coexistence equilibrium point can be simplify at following form:

$$\mathbf{J}|_{(\mathbf{x}^*,\mathbf{y}^*)} = \begin{pmatrix} (1-rx^*) & -ax^* \\ \frac{y^*}{x^*}(1-rx^*) & ax^* \end{pmatrix}.$$

For the coexistence equilibrium point we have,

$$\begin{cases} det(\mathbf{J}\mid_{(\mathbf{x}^*,\mathbf{y}^*)}) = x^*a - x^{*2}ar + ay^* - ay^*rx^*, \\ tr(\mathbf{J}\mid_{(\mathbf{x}^*,\mathbf{y}^*)}) = 1 - rx^* + x^*a. \end{cases}$$

The eigenvalues are,

$$\lambda_1 = \frac{1}{2}xa + \frac{1}{2} - \frac{1}{2}xr + \frac{1}{2}\sqrt{x^2a^2 + 2xa - 2x^2ar + 1 - 2xr + x^2r^2 - 4xae^{ay} + 4x^2are^{ay}},$$

$$\lambda_2 = \frac{1}{2}xa + \frac{1}{2} - \frac{1}{2}xr - \frac{1}{2}\sqrt{x^2a^2 + 2xa - 2x^2ar + 1 - 2xr + x^2r^2 - 4xae^{ay} + 4x^2are^{ay}}.$$

Note. The eigenvalues are complex conjugate if [5]

$$det\mathbf{J} > (\frac{tr\mathbf{J}}{2})^2 \quad \Longrightarrow \quad x^{*2}ar + ay^* - ay^*rx^* > (\frac{1 - rx^* + x^*a}{2})^2$$

then the dynamical system is characterized by:

- Spiral Sink, if and only if,

$$det\mathbf{J} < 1 \quad \Longrightarrow \quad x^*a - x^{*2}ar + ay^* - ay^*rx^* < 1.$$

- Spiral Source, if and only if,

$$det\mathbf{J} > 1 \quad \Longrightarrow \quad x^*a - x^{*2}ar + ay^* - ay^*rx^* > 1.$$

Theorem 2.3. *The coexistence equilibrium point is globally stable and convergence is spiral if and only if,* $x^*a - x^{*2}ar + ay^* - ay^*rx^* < 1$, *whereas the steady-state equilibrium is unstable and divergence is spiral if and only if,* $x^*a - x^{*2}ar + ay^* - ay^*rx^* > 1$ *[5].*

Theorem 2.4. *The coexistence equilibrium point in the system (1) is asymptotically stable if we have:*

$$|1 - rx^* + x^*a| < 1 + x^*a - x^{*2}ar + ay^* - ay^*rx^* < 2.$$

In Fig. 1, we see the results simulation for different values of a for system (1). This figure shows the limit cycles of herbivore and plant. When the parameter a increase, this asymptotically stable equilibrium loses its stability and rise to an invariant closed curve. The dynamics of the invariant curve include periodic and quasi-periodic orbits. Eventually, the invariant curve loses its smoothness and breaks up into a chaotic attractor [6].

Theorem 2.5. *A general two-dimensional map has a stable and an unstable manifolds when one of the following conditions, in the Trace-Determinant plane, are satisfied [7]*

$$\begin{cases} i) det(J^*) < tr(J^*) - 1 \quad , \quad det(J^*) > -tr(J^*) - 1, \\ ii) det(J^*) > tr(J^*) - 1 \quad , \quad det(J^*) < -tr(J^*) - 1, \\ iii) det(J^*) < \frac{(tr(J^*))^2}{4} \quad , \quad det(J^*) > 1 \quad , \quad det(J^*) > -tr(J^*) - 1, \\ iv) det(J^*) < \frac{(tr(J^*))^2}{4} \quad , \quad det(J^*) > tr(J^*) - 1 \quad , \quad det(J^*) > 1. \end{cases}$$

Theorem 2.6. *For axial fixed point* $(1, 0)$ *of system (1) we have a stable and an unstable manifolds, when one of the following conditions are satisfied:*

i) $a(1 - r) < a - r, \quad a(1 - r) > r - a - 2,$

ii) $a(1 - r) > a - r, \quad a(1 - r) < r - a - 2,$

iii) $a(1 - r) < \frac{(1-r+a)^2}{4}, \quad a(1 - r) > 1, \quad a(1 - r) > r - a - 2,$

iv) $a(1 - r) < \frac{(1-r+a)^2}{4}, \quad a(1 - r) > a - r, \quad a(1 - r) > 1.$

Theorem 2.7. *For coexistence equilibrium point of system (1) we have a stable and an unstable manifolds, when one of the following conditions are satisfied:*

i) $x^*a - x^{*2}ar + ay^* - ay^*rx^* < -rx^* + x^*a,$
$x^*a - x^{*2}ar + ay^* - ay^*rx^* > -2 + rx^* - x^*a,$

ii) $x^*a - x^{*2}ar + ay^* - ay^*rx^* > -rx^* + x^*a,$
$x^*a - x^{*2}ar + ay^* - ay^*rx^* < -2 + rx^* - x^*a,$

iii) $x^*a - x^{*2}ar + ay^* - ay^*rx^* < \frac{(1-rx^*+x^*a)^2}{4},$
$x^*a - x^{*2}ar + ay^* - ay^*rx^* > 1,$
$x^*a - x^{*2}ar + ay^* - ay^*rx^* > -2 + rx^* - x^*a,$

iv) $x^*a - x^{*2}ar + ay^* - ay^*rx^* < \frac{(1-rx^*+x^*a)^2}{4},$
$x^*a - x^{*2}ar + ay^* - ay^*rx^* > -rx^* + x^*a,$
$x^*a - x^{*2}ar + ay^* - ay^*rx^* > 1,$

In Fig. 2, we see the time series for different values of a.

3. The bifurcations

Let $x_{t+1} = G(x_t; \mu)$ be a general one-dimensional family of map, where $x_t \in \mathbb{R}$ and $\mu \in \mathbb{R}$, and, if μ_c indicates a value of controlling parameter, let $\bar{x}(\mu_c)$ be a corresponding equilibrium value [8].

Remark. 1 To detect the local bifurcations we use the following conditions [8]:

1) $\frac{\partial G(\bar{x};\mu_c)}{\partial x_t} = 1$, simultaneously for fold, transcritical and pitchfork;

2) $\frac{\partial^2 G(\bar{x};\mu_c)}{\partial x_t^2} \neq 0$, simultaneously for fold and transcritical;

3) $\frac{\partial^2 G(\bar{x};\mu_c)}{\partial x_t^2} = 0$ and $\frac{\partial^3 G(\bar{x};\mu_c)}{\partial x_t^3} \neq 0$, for pitchfork;

4) $\frac{\partial G(\bar{x};\mu_c)}{\partial \mu} \neq 0$, for fold;

5) $\frac{\partial G(\bar{x};\mu_c)}{\partial \mu} = 0$ and $\frac{\partial^2 G(\bar{x};\mu_c)}{\partial x_t \partial \mu} \neq 0$, simultaneously for transcritical and pitchfork.

We assume that,

$$G(\bar{v}) = a\bar{v} + a^2\bar{v}^2(\frac{1}{2!} - \frac{1}{r}) + a^3\bar{v}^3(\frac{1}{3!} - \frac{1}{2!}\frac{1}{r}),$$

We have,

$$\begin{aligned}
\frac{\partial G(\bar{v})}{\partial v} &= a + 2a^2\bar{v}(\frac{1}{2} - \frac{1}{r}) + 3a^3\bar{v}^2(\frac{1}{6} - \frac{1}{2r}), \\
\frac{\partial^2 G(\bar{v})}{\partial v^2} &= 2a^2(\frac{1}{2} - \frac{1}{r}) + 6a^3\bar{v}(\frac{1}{6} - \frac{1}{2r}), \\
\frac{\partial^3 G(\bar{v})}{\partial v^3} &= 6a^3(\frac{1}{6} - \frac{1}{2r}), \\
\frac{\partial G(\bar{v})}{\partial a} &= \bar{v} + 2a\bar{v}^2(\frac{1}{2} - \frac{1}{r}) + 3a^2\bar{v}^3(\frac{1}{6} - \frac{1}{2r}), \\
\frac{\partial G(\bar{v})}{\partial r} &= \frac{a^2\bar{v}^2}{r^2} + \frac{1}{2}\frac{a^3\bar{v}^3}{r^2}, \\
\frac{\partial^2 G(\bar{v})}{\partial v \partial a} &= 1 + 4a\bar{v}(\frac{1}{2} - \frac{1}{r}) + 9a^2\bar{v}^2(\frac{1}{6} - \frac{1}{2r}), \\
\frac{\partial^2 G(\bar{v})}{\partial v \partial r} &= \frac{2a^2\bar{v}}{r^2} + \frac{3}{2}\frac{a^3\bar{v}^2}{r^2}.
\end{aligned}$$

Proposition. 1 For eq. (2), at $a = 1$, $r \neq 2$ and $\bar{v}_1$ we have (without herbivore),

1) $\frac{\partial G(\bar{v})}{\partial v} = 1$, simultaneously for fold, transcritical and pitchfork;

2) $\frac{\partial^2 G(\bar{v})}{\partial v^2} \neq 0$, simultaneously for fold and transcritical;

3) $\frac{\partial G(\bar{v})}{\partial a} = 0$ and $\frac{\partial^2 G(\bar{v})}{\partial v \partial a} \neq 0$, simultaneously for transcritical and pitchfork.

Proposition. 2 For eq. (2), at $a \neq 1$, $r = 2$ and $\bar{v}_1$ we have (without herbivore),

1) $\frac{\partial^2 G(\bar{v})}{\partial v^2} = 0$ and $\frac{\partial^3 G(\bar{v})}{\partial v^3} \neq 0$, for pitchfork;

Proposition. 3 For eq. (2), at $a = 1$, $r = 2$ and $\bar{v}_1$ we have (without herbivore),

1) $\frac{\partial G(\bar{x})}{\partial v} = 1$, simultaneously for fold, transcritical and pitchfork;

2) $\frac{\partial^2 G(\bar{v})}{\partial v^2} = 0$ and $\frac{\partial^3 G(\bar{v})}{\partial v^3} \neq 0$, for pitchfork;

Corollary. 1 For eq. (2), at $a = 1$, $r \neq 2$ and $\bar{v}_3$ we have,

1) $\frac{\partial^2 G(\bar{v})}{\partial v^2} \neq 0$, simultaneously for fold and transcritical;

2) $\frac{\partial G(\bar{v})}{\partial a} \neq 0$, $\frac{\partial G(\bar{v})}{\partial r} \neq 0$ for fold;

Corollary. 2 For eq. (2), at $a \neq 1$, $r = 2$ and $\bar{v}_3$ we have,

1) $\frac{\partial^2 G(\bar{v})}{\partial v^2} \neq 0$, simultaneously for fold and transcritical;

2) $\frac{\partial G(\bar{v})}{\partial a} \neq 0$, $\frac{\partial G(\bar{v})}{\partial r} \neq 0$ for fold;

Corollary. 3 For eq. (2), at $a = 1$, $r = 2$ and $\bar{v}_3$ we have,

1) $\frac{\partial G(\bar{v})}{\partial v} = 1$, simultaneously for fold, transcritical and pitchfork;

2) $\frac{\partial^2 G(\bar{v})}{\partial v^2} = 0$ and $\frac{\partial^3 G(\bar{v})}{\partial v^3} \neq 0$, for pitchfork;

3) $\frac{\partial G(\bar{v})}{\partial a} = 0$ and $\frac{\partial^2 G(\bar{v})}{\partial v \partial a} \neq 0$, simultaneously for transcritical and pitchfork.

Theorem 3.1. *The saddle-node bifurcation occurs when the Jacobian matrix has an eigenvalue equal to 1. This is equivalent to saying that we cross the line $det(J^*) = tr(J^*) - 1$ from the stability region [7].*

Note. The saddle-node bifurcation is the type I intermittency[2](route to chaos). Hilborn has discussed about types of intermittency in his book [9].

Theorem 3.2. *The period-doubling bifurcation occurs when the Jacobian matrix has an eigenvalue equal to -1.*
In the T-D plane this occurs as we cross the line $det(J^*) = -tr(J^*) - 1$ *from the stability region [7].*

Result 1. If $a = 1$ at $(1, 0)$, for system (1) the saddle-node bifurcation occurs.

Result 2. For the axial fixed point $(1, 0)$, $\lambda_2 = 1 - r = -1$, This leads $r = 2$. In this case for system (1) the period-doubling bifurcation occurs.

Here, we study the Neimark-Sacker bifurcation for the coexistence equilibrium point of system (1).
Neimark (1959) and Sacker (1965) stated relevant results about the case in which a pair of complex eigenvalues of the Jacobian matrix at the fixed point of a discrete map has modulus one [8].

Theorem 3.3. *When* $det(J^*) = 1$ *and* $|tr(J^*)| < 2$*, Neimark-Sacker bifurcation occurs [7].*

Result 3. For occurrence the Neimark-sacker bifurcation we must be have,

$$\begin{cases} det(J^*) = 1 & \Longrightarrow \quad x^*a - x^{*2}ar + ay^* - ay^*rx^* = 1, \\ |tr(J^*)| < 2 & \Longrightarrow \quad |1 - rx^* + x^*a| < 2 \end{cases}$$

For $det(J^*) = 1$, we must be have,

$$x_1 = \frac{1}{2}\frac{ae^{ay} + \sqrt{a^2(e^{ay})^2 - 4are^{ay}}}{are^{ay}}, \quad x_2 = \frac{1}{2}\frac{ae^{ay} - \sqrt{a^2(e^{ay})^2 - 4are^{ay}}}{are^{ay}},$$

$$y = \frac{ln(\frac{1}{ax(1-xr)})}{a}.$$

Note. If the two floquet multipliers To make a complex conjugate pair, we have the Hopf (Neimark-Sacker) bifurcation. The Hopf

[2] In intermittency, the behaviour of system switch between periodic, quasi periodic and chaotic.

(Neimark-Sacker) bifurcation is the type II intermittency. Type II intermittency has been observed, in only a few experimental studies [9].
In Fig. 3-(b) , we see invariant loops for $2.61 < r < 2.64$ and $a = 1.5$.
In Fig. 3-(c) , we see period 21 versus chaotic orbits and in Fig. 3-(d) , we see period 6 versus chaotic orbits. This picture show the bistability dynamics that are analysed by Yun Kang, Dieter Armbruster and Yang Kuang in 2008 [1]. The defining properties of bistability is two stable states are separated by an unstable state.
For $a = 1.5$, when the parameter r is changed, the asymptotically stable fixed point loses its stability and to be an attracting closed invariant curve gradually. When the values of r increase, radius of the curve enhance.

4. The case $y_t = 0$

Dynamics of this system for $y_t = 0$ is similar to the dynamics which is given by the ricker curve $x_{t+1} = x_t e^{r(1-x_t)}$. The period-doubling bifurcations, periodic windows and chaos are observed in the ricker curve [3]. In this case, $x = 1$ is a steady state.

$$x_{n+1} = x_n e^{r(1-x_n)} \Rightarrow H_r(x) = xe^{r(1-x)} \Rightarrow H_r(x^*) = x^* \Rightarrow x^* = 1.$$

Here, we study the stability and bifurcation in this case.

$$\frac{\partial H}{\partial x}(r,x) = e^{r(1-x)} - rxe^{r(1-x)} \Longrightarrow \frac{\partial H}{\partial x}(r^*,x^*) = 1-r^* \Longrightarrow |H_r'(x^*)| < 1$$

$$\Longleftrightarrow 0 < r < 2.$$

A stable coexistence is observed when $r < 2$. The chaotic attractor appear when r is approximately increased from 2.7 . Here, complex dynamics patterns have been observed.
For $r = 2$, we have,

$$\frac{\partial H}{\partial x}(r^*,x^*) = -1, \quad \frac{\partial^2 H^2}{\partial r \partial x}(r^*,x^*) = 3 \neq 0.$$

In this case, the period-doubling bifurcation occurs.
for $r = 2$, $x = 1$ is asymptotically stable.

$$\mathbf{S}H(x^*) = \frac{H'''(x^*)}{H'(x^*)} - \frac{3}{2}[\frac{H''(x^*)}{H'(x^*)}]^2 = -4 < 0.$$

For $r > 3.0$, extinction in this model is obviously.

5. Adding an immigration term on plant dynamic

In this section, we study the effects of added immigration on model (1). Chaos in high level of immigration vanishes. But, in low level of immigration, the chaos behaviours persist.
In [10], we can see, Chaos reached via the quasiperiodic route is more robust against the perturbation than period-doubling chaos.
To investigate the robustness of the chaotic dynamics in model (1) we analyse the effects to the plant population,

$$\begin{cases} x_{t+1} = x_t e^{r[1-x_t]-ay_t} + \varepsilon x_t \\ y_{t+1} = x_t e^{r[1-x_t]}[1-e^{-ay_t}], \end{cases} \tag{5.1}$$

where ε is the percentage immigration term.
In model (3), we have one extinction fixed point (0,0) and one coexistence fixed point.
The Jacobian matrix for system (3), has the following form:

$$\mathbf{J} = \begin{pmatrix} e^{r(1-x)-ay}(1-rx)+\varepsilon & -axe^{r(1-x)-ay} \\ e^{r(1-x)}(1-e^{-ay})(1-rx) & axe^{r(1-x)-ay} \end{pmatrix}.$$

Jacobian matrix at $(0,0)$ is calculated:

$$\mathbf{J}\mid_{(\mathbf{0},\mathbf{0})} = \begin{pmatrix} e^r+\varepsilon & 0 \\ 0 & 0 \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = e^r + \varepsilon$ and $\lambda_2 = 0$. Since $r > 0$, and hence $e^r > 1$, the eigenvalue $e^r + \varepsilon$ is greater than 1 that corresponds to a one-dimensional unstable manifold and $\lambda_2 = 0$ correspond to a one-dimensional stable manifold.

Theorem 5.1. *The extinction fixed point (0,0) of the system (3) is a saddle point and unstable.*

The coexistence equilibriums (x^*, y^*) for system (3) is,

$$\begin{cases} x^* = \frac{r-ln(1-\varepsilon)-ay^*}{r}, \\ \\ y^* = \frac{r-ln(1-\varepsilon)-ay^*}{r}(1-\varepsilon)(1-e^{-ay^*}). \end{cases}$$

The Jacobian at the coexistence equilibrium point can be simplify at following form:

$$\mathbf{J}|_{(\mathbf{x}^*,\mathbf{y}^*)} = \begin{pmatrix} (1-rx^*)(1-\varepsilon) & -ax^*(1-\varepsilon) \\ \frac{y^*}{x^*}(1-rx^*) & ax^*(1-\varepsilon) \end{pmatrix}.$$

For the coexistence equilibrium point we have,

$$\begin{cases} det(\mathbf{J}\mid_{(\mathbf{x}^*,\mathbf{y}^*)}) = y^*a(1-\varepsilon-rx^*+rx^*\varepsilon)+x^*a(1-rx^*-2\varepsilon+2x^*\varepsilon r+\varepsilon^2-x^*\varepsilon^2 r), \\ tr(\mathbf{J}\mid_{(\mathbf{x}^*,\mathbf{y}^*)}) = (1-rx^*)(1-\varepsilon)+x^*a(1-\varepsilon). \end{cases}$$

It can be seen in Fig. 4, that a 0.5 immigration level prevents the emergence of chaos via the period-doubling route. For 0.3 immigration level , the period-doubling route to chaos is inhibited.

6. Main result

In this paper, we studied a discrete-time plant-herbivore model. We investigated stability of model. The equilibrium solutions and the local asymptotic stability are analysed. We saw the different behaviour from regular to chaos.

Bifurcation analysis done with respect to intrinsic growth rate r. Many forms of complex dynamics such as chaos, periodic windows etc. were observed. Transition routes to chaos dynamics were established via period-doubling bifurcation. The conditions of occurrence the period-doubling, Neimark-Sacker and saddle-node bifurcations were analysed. Type I intermittency (Neimark-Sacker bifurcation) and type II intermittency (saddle-node bifurcation) were observed. We implied conditions that the general two-dimensional map has a stable and an unstable manifolds and studied the stable and unstable manifolds for this system in equilibrium points. For $y_t = 0$, this system was similar to the dynamics of ricker curve.

We added an immigration term to plant equation and we noticed a high immigration level prevented the emergence of chaos via the period-doubling route. For low immigration level , the period-doubling route to chaos was inhibited.

For the different values of r, the host x_t and parasitoid y_t approached to fixed values. As r was changed, the dynamics on the invariant curve exhibited periodic and quasi-periodic orbits. Furthermore, the invariant closed curves lost its stability and broke up into chaotic attractors.

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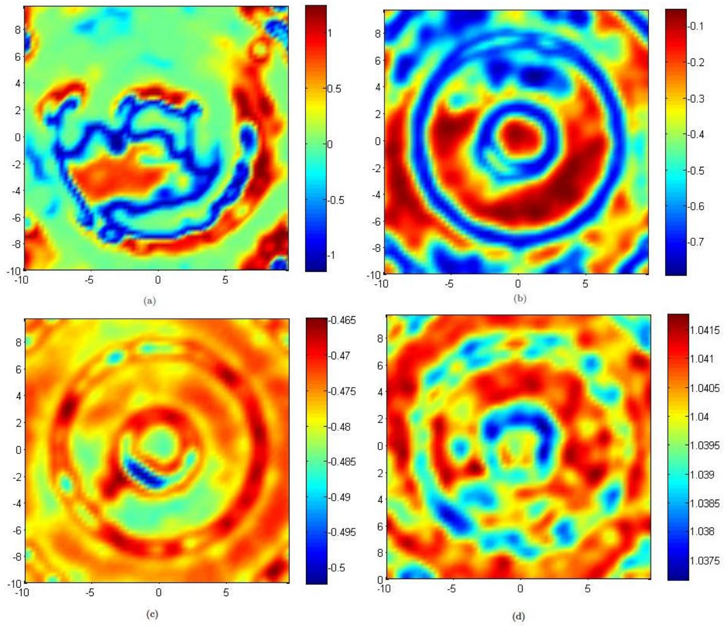

FIGURE 1. Phase portrait for system (1) with r = 1.75 and initial values $x_0 = 0.1$, $y_0 = 0.9$, (a) a = 4, (b) a = 2, (c) a = 1.5, (d) a = 0.5.

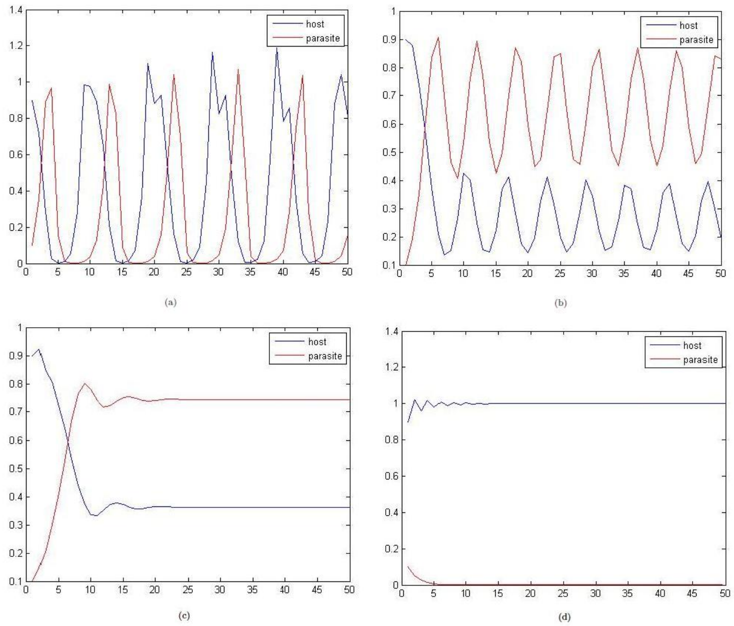

FIGURE 2. Phase portrait for system (1) with $r = 1.75$ and initial values $x_0 = 0.1$, $y_0 = 0.9$, (a) a = 4, (b) a = 2, (c) a = 1.5, (d) a = 0.5.

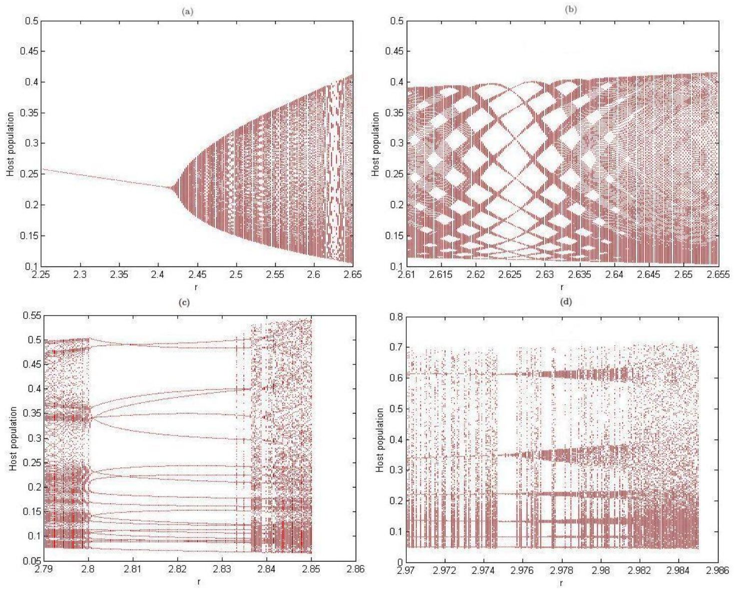

FIGURE 3. (a) Bifurcation diagram for system (1), (b) invariant loops, (c) period 21 versus chaotic orbits, (d) period 6 versus chaotic orbits.

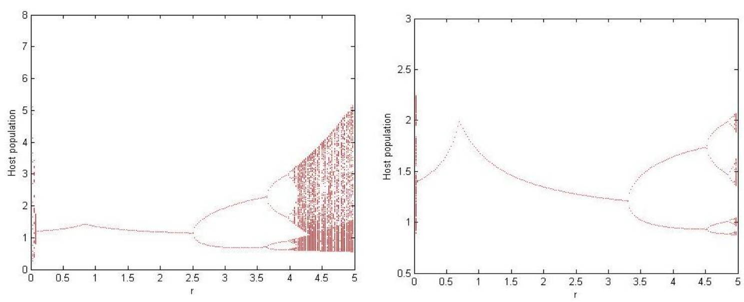

FIGURE 4. Bifurcation diagram for system (3) with $a = 1$, initial values $x_0 = 0.2$, $y_0 = 0.78$, and $\varepsilon = 0.3$ (left), $\varepsilon = 0.5$ (right).