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The Efficiency of Harvested Factor; Lotka-Volterra Predator-Prey Model

M. H. Rahmani Doust [1]
Department of Mathematics, University of Neyshabur, Neyshabur, Iran

Abstract. Scientists are interested in find out "how to use living resources without damaging the ecosystem at the same time?" from nineteen century because the living resources are limited. Thus, the harvested rate is used as the control parameters. Moreover, the study of harvested population dynamics is more realistic. In the present paper, some predator-prey models in which two ecologically interacting species are harvested independently with constant or variable rates have been considered. Also, the behavior of their solutions in the global and local stability aspect have been investigated. The main aim is to present a mathematical analysis for the above model.

Keywords: Equilibrium Point, Lotka-Volterra Model, Predator-Prey System, Stability.

2000 Mathematics subject classification: 34C30, 34D45, 92D25.

1. Introduction

The problem of predator-prey is well-known and an old problem which was first introduced by A.J. Lotka [4] and V. Volterra [11] independently around the last century. The standard Lotka-Volterra system has been widely investigated [2,3,5,6]. Considering situation that j^{th}-population may affect upon to the i^{th}-population linearly, one can find following system. That is the effect of j^{th}-population upon to the i^{th}-population may be shown by a_{ij} which is constant real number. Indeed, by making

[1] mh.rahmanidoust@neyshabur.ac.ir
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the above assumption, the general form lotka- Volterra equations may be found as follow:

$$\frac{dx_i}{dt} = x_i\Big(r_i + \sum_{j=1}^{n} a_{ij}x_j\Big), \qquad i = 1, 2, 3, ..., n; \tag{1.1}$$

known as multi-species Lotka-Volterra model, in which it may distinguish into three cases: predator-prey, competition and coexistence system which some of them has been worked out in [7,8,9,10]. A globally dynamics for a harvested predator-prey system has been analyzed in [1]. Not that it is assumed that all parameters $a, b, c, d, h, i, v, e, p, q, \alpha, \beta, \gamma_1, \gamma_2$ are positive constant.

2. Analysis of Harvested Predator-Prey Systems (Simple Models)

2.1. Predator-Prey Model Having Constant Harvesting Factors.

Consider Lotka-Volterra predator-prey model which has constant harvest factor for prey and predator species.

$$\begin{cases} \frac{dx}{dt} = x(a - by) - h, \\ \\ \frac{dy}{dt} = y(-c + dx) - i, \end{cases} \tag{2.1}$$

where a, c are growth rates of prey and predator respectively. The coefficient b, d are efficiency of predator on prey and prey on predator respectively and h, i are constant harvest factors.

Theorem 2.1. *Let (x^*, y^*) ia an equilibrium point for system (2.1). This system is unstable at its equilibrium point. Therefor, the proof is done.*

Proof. The equilibrium point of the system (2.1) is

$$(x^*, y^*) = (\frac{h}{a - by^*}, \frac{(ac - ib) + \sqrt{(ac - ib)^2 + 4bc(ia - dh)}}{2bc})$$

where $\frac{i}{h} > \frac{d}{a}$.

Now define Lyapunov function

$$V(x, y) = (x - x^*) - x^* Ln\frac{x}{x^*} + \frac{b}{d}[(y - y^*) - y^* Ln\frac{y}{y^*}].$$

By differentiation of Lyapunov function with respect to variable t we have

$$\frac{dV}{dt}(x, y) = h\frac{(x - x^*)^2}{xx^*} + \frac{bi}{d}\frac{(y - y^*)^2}{yy^*}.$$

Since $\frac{dV}{dt} > 0$, The system (2.1) is unstable. □

One of the special cases of the system (2.1) is

$$\begin{cases} \frac{dx}{dt} = x(a - by) - h, \\ \\ \frac{dy}{dt} = y(-c + dx). \end{cases} \tag{2.2}$$

Which is known as predator-prey model having constant prey harvesting. We will analyze local stability of the above system. The equilibrium points of system (2.2) are $(\frac{h}{a}, 0)$, $(\frac{c}{d}, \frac{a}{b} - \frac{dh}{cb})$. Since the eigenvalues corresponding to the equilibrium point $(\frac{h}{a}, 0)$ are $\lambda_1 = a$ and $\lambda_2 = -c$, and $\lambda_1 \lambda_2 < 0$ it is a saddle point.

For the second equilibrium point the eigenvalues are

$$\frac{\frac{dh}{c} \pm \sqrt{(\frac{dh}{c})^2 - 4(ac - dh)}}{2}.$$

Thus the said point is unstable.

The second specially case of model (2.1) is

$$\begin{cases} \frac{dx}{dt} = x(a - by), \\ \\ \frac{dy}{dt} = y(-c + dx) - i. \end{cases} \tag{2.3}$$

It is known as predator-prey model having constant predator harvest factor. The above model has one equilibrium point $(\frac{ib+ac}{d}, \frac{a}{b})$. If $ib+ac > c$, then the said equilibrium point is unstable point for system (2.3).

2.2. **Predator- Prey Model Having Variable Harvesting Factor.** Consider the following model, which has effort rate, corresponding to the population densities. In the following model e_1, e_2 are positive constants that show effort rate to tong prey and predator respectively.

$$\begin{cases} \frac{dx}{dt} = x(a - by) - e_1 x, \\ \\ \frac{dy}{dt} = y(-c + dx) - e_2 y. \end{cases} \tag{2.4}$$

The above model with making variations $a - e_1$ and $c + e_2$ into a and c respectively can be transferred into the model that, if $a > e_1$ is the simplest ordinary predator-prey system. Two special cases of the system(2.4) are called as predator-prey model having prey harvesting (corresponding to prey population) and predator-prey model having predator harvesting (corresponding to predator population), which may be respectively written as follows

$$\begin{cases} \frac{dx}{dt} = x(a - by) - ex, \\ \\ \frac{dy}{dt} = y(-c + dx). \end{cases} \tag{2.5}$$

$$\begin{cases} \frac{dx}{dt} = x(a - by), \\ \\ \frac{dy}{dt} = y(-c + dx) - ey. \end{cases} \tag{2.6}$$

Analysis of three above systems are clear.

2.3. Predator-Prey Model Having Variable harvesting Factor(Inverse Ratio).

$$\begin{cases} \frac{dx}{dt} = x(a - by) - \frac{p}{x} \\ \\ \frac{dy}{dt} = y(-c + dx) - \frac{q}{y} \end{cases} \tag{2.7}$$

Theorem 2.2. *The system (2.7) is unstable at its equilibrium point.*

Proof. Let $(\overline{x}, \overline{y})$ be it's equilibrium point.

Defining Lyapunov function

$$V(x, y) = \int_{\overline{x}}^{x} \frac{s - \overline{x}}{s} ds + \frac{b}{d} \int_{\overline{y}}^{y} \frac{t - \overline{y}}{t} dt$$

implies that

$$\frac{dV}{dt} = -p(x - \overline{x})(\frac{1}{\overline{x}^2} - \frac{1}{\overline{x}^2}) - \frac{bq}{d}(y - \overline{y})(\frac{1}{y^2} - \frac{1}{\overline{y}^2}) = p\frac{(x - \overline{x})^2(x + \overline{x})}{(x\overline{x})^2} + \frac{bq(y - \overline{y})^2(y + \overline{y})}{d(y\overline{y}^2)}.$$

It is clear that $\frac{dV}{dt} > 0$, and so the proof is done. □

A special case of system (2.7) is

$$\begin{cases} \frac{dx}{dt} = x(a - by) - \frac{h}{x}, \\ \\ \frac{dy}{dt} = y(-c + dx). \end{cases} \tag{2.8}$$

which is known as predator-prey model having variable prey harvesting(inverse ratio of prey population).

Theorem 2.3. *For system (2.8) the following statements are true:*

i) If $\frac{a}{h} < \frac{d^2}{c^2}$, then (x^*, y^*) is a saddle point.

ii) If $\frac{d^2}{c^2} < \frac{a}{h} < \frac{d^2}{c^2}(1 + \frac{1}{2c})$, then the above equilibrium point is unstable.

iii) If $\frac{a}{h} > \frac{d^2}{c^2}(1 + \frac{1}{2c})$, then the above equilibrium point is a spiral point.

Proof. Its equilibrium points in the first phase quadratic are

$$(\overline{x}, \overline{y}) = (\sqrt{\frac{h}{a}}, o), (x^*, y^*) = (\frac{c}{d}, \frac{a}{b} - \frac{hd^2}{bc^2}).$$

The Jacobian matrix at the point $(\overline{x}, \overline{y})$ may be found as the follow

$$J|_{(\overline{x},\overline{y})} = \begin{pmatrix} 2a & -b\sqrt{\frac{h}{a}} \\ 0 & -c + d\sqrt{\frac{h}{a}} \end{pmatrix}$$

Hence, its eigenvalues are $\lambda_1 = 2a$ and $\lambda_2 = -c + d\sqrt{\frac{h}{a}}$. Thus, the related point is saddle provided $\sqrt{\frac{h}{a}} < \frac{c}{d}$. And for the second point (x^*, y^*), Jacobian matrix is as follow

$$J|_{(x^*,y^*)} = \begin{pmatrix} \frac{2hd^2}{c^2} & -\frac{bc}{d} \\ \frac{d}{b}(a - \frac{hd^2}{c^2}) & 0 \end{pmatrix}$$

Thus, the proof is done. □

3. Analysis of Harvested Predator-Prey Systems (Complex Models)

3.1. **Complex Harvested Model (1).** Let us consider diseases exists in a ecosystem or there is interspecific competition for the predator-prey model which may be written as follow

$$\begin{cases} \frac{dx}{dt} = x(a - by) - \beta x^2 \\ \\ \frac{dy}{dt} = y(-c + dx) - \alpha y^2 \end{cases} \tag{3.1}$$

The linearization method may be applied in the above model. The equilibrium point $(0, 0)$ is saddle, this system at the point $(0, -\frac{c}{\alpha})$ is stable if $\frac{c}{d} > \frac{a}{\beta}$ this system is asymptotically stable in $(\frac{a}{\beta}, 0)$. And finally, the last equilibrium point denoted by $(\overline{x}, \overline{y})$

where

$a - b\overline{y} - \beta\overline{x} = 0 \ , \ -c + d\overline{x} - \alpha\overline{y} = 0.$

$\Rightarrow J|_{(\overline{x},\overline{y})} = \begin{pmatrix} a - b\overline{y} - 2\beta\overline{x} & -b\overline{x} \\ d\overline{y} & -c + d\overline{x} - 2\alpha\overline{y} \end{pmatrix} = \begin{pmatrix} -\beta\overline{x} & -b\overline{x} \\ d\overline{y} & -\alpha\overline{y} \end{pmatrix}$

$\Rightarrow \lambda^2 - \lambda(\beta\overline{x} + \alpha\overline{y}) + \alpha\beta\overline{xy} + bd\overline{xy} = 0$

Regarding the $\beta\overline{x} + \alpha\overline{y}) > 0$ and $\alpha\beta\overline{xy} + bd\overline{xy} > 0$, by using Hurwitz theorem the last equation has two negative roots and the system (3.1) is locally asymptotically stable. One can see more information about system (3.1) in [2,3,5,7,9].

3.2. **Complex Harvested Model(2).** In this subsection we will mix two harvest factors for prey and predator species. In the following model the prey species has effort rate harvest factor, and The predator has a complex harvest factor with respect to density of itself and inverse ratio of the density of prey.

$$\begin{cases} \frac{dx}{dt} = x(a - by) - \beta x, \\ \\ \frac{dy}{dt} = y(-c + dx) - \frac{y}{vx}, \end{cases} \tag{3.2}$$

where β is effort rate for prey species and v is number of needful preys to support one predator. The above system has just one equilibrium point

$$(x^*, y^*) = (\frac{cv + \sqrt{(cv^2) + 4dv}}{2dv}, \frac{a - \beta}{b}).$$

By using linearization method one can see that that the above system is stable provided $a \geq \beta$. Indeed, the following theorem has been proved:

Theorem 3.1. *The equilibrium point (x^*, y^*) of system (3.2) is stable provided $a \geq \beta$.*

3.3. **Complex Harvested Model (3).** Now similarly to above section we will mix two harvest factors. For the prey species competition and for the predator species a complex harvest factor with respect to density of itself and inverse ratio of the density of prey.

$$\begin{cases} \frac{dx}{dt} = x(a - by) - \beta x^2, \\ \\ \frac{dy}{dt} = y(-c + dx) - \frac{y}{vx}. \end{cases} \tag{3.3}$$

Theorem 3.2. *i. The system (3.3) has two equilibrium points.*
ii. The equilibrium point $(\frac{a}{\beta}, 0)$ locally asymptotically stable provided $\frac{ad}{\beta} < c + \frac{\beta}{av}$.
iii. The system (3.3) is locally asymptotically stable at its other equilibrium point.

Proof. One of the its equilibrium point is $(\frac{a}{\beta}, 0)$ and the other is shown by (x^*, y^*). Linearization method shows its eigenvalues at equilibrium point $(\frac{a}{\beta}, 0)$ are $\lambda_1 = -a$ and $\lambda_2 = -c + \frac{ad}{\beta} - \frac{\beta}{av}$ and so,the said point is locally asymptotically stable provided $\frac{ad}{\beta} < c + \frac{\beta}{av}$. For the second equilibrium point we have

$$A = J|_{(x^*, y^*)} = \begin{pmatrix} -\beta x^* & -bx^* \\ \frac{d}{b}(a - \beta x^*) + \frac{a - \beta x^*}{bvx^{*2}} & 0 \end{pmatrix}$$

Thus $tracA < 0$ and $detA > 0$, so related system has two eigenvalue has distinct negative real parts. Hence, the proof is done. $\square$

3.4. **Complex Harvested Model(4).** Consider

$$\begin{cases} \frac{dx}{dt} = x(a - by) - \frac{x}{y}, \\ \\ \frac{dy}{dt} = y(-c + dx) - \frac{y}{x+q}. \end{cases} \tag{3.4}$$

Which is harvested by complex harvested factors for prey and predator species. These harvest factors are congruent by density of each species and inverse ratio of density of other species. Related the equilibrium points are given by $(\overline{x}, \overline{y})$ and $(\overline{x}, y^*)$ where

$$\overline{x} = \frac{(c - dq) + \sqrt{(dq - c)^2 + 4d(cq + 1)}}{2d},$$

$$\overline{y} = y^* = \frac{a + \sqrt{a^2 - 4b}}{2b}$$

A simple calculation shows that the equilibrium point $(\overline{x}, \overline{y})$ is asymptotically stable provided that parameters A and B be negative, where

$$A = a - b\overline{y} - \frac{1}{\overline{y}} - c + d\overline{x} - \frac{1}{q + \overline{x}},$$

$$B = (a - b\overline{y} - \frac{1}{\overline{y}})(-c + d\overline{x} - \frac{1}{q + \overline{x}}) - (d\overline{y} + \frac{\overline{y}}{(\overline{x} + q)^2})(-b\overline{x} + \frac{\overline{x}}{\overline{y}^2}).$$

Similarly for the second equilibrium point $(\overline{x}, y^*)$, we will find

$$\lambda_1, \lambda_2 = \frac{A_1 \pm \sqrt{A_1^2 - 4B_1}}{2}$$

where $A_1 = a - by^* - \frac{1}{y^*} - c + d\overline{x} - \frac{1}{q+\overline{x}}$,
$B_1 = -(a - by^* - \frac{1}{y^*})(-c + d\overline{x} - \frac{1}{q+\overline{x}}) + (dy^* + \frac{y^*}{(\overline{x}+q)^2})(-b\overline{x} + \frac{\overline{x}}{y^{*2}})$.

3.5. **Complex Harvested Model(5).** In this case we are willing to study efficiency of greater harvest factors with respect to ago models.

$$\begin{cases} \frac{dx}{dt} = x(a - by) - \beta x^3, \\ \\ \frac{dy}{dt} = y(-c + dx) - \frac{y^2}{vx}. \end{cases} \tag{3.5}$$

The equilibrium points for the above system are given by

$$(\sqrt{\frac{a}{\beta}}, 0)$$

and

$$(\overline{x}, \overline{y}),$$

where

$$\overline{x} = \frac{bcv + \sqrt{(bcv)^2 + 4a(bdv + \beta)}}{2(bdv + \beta)}$$

$$\overline{y} = v\overline{x}(-c + d\overline{x}).$$

Related eigenvalues for above system are $\lambda_1 = -2a$ and $\lambda_2 = -c + d\sqrt{\frac{a}{\beta}}$ at equilibrium point $(\sqrt{\frac{a}{\beta}}, 0)$. Simple calculating proves the following theorem:

Theorem 3.3. *The solution of system (3.5) at the point* $(\sqrt{\frac{a}{\beta}}, 0)$ *is stable provided* $\frac{c}{d} > \sqrt{\frac{a}{\beta}}$.

For the second equilibrium point, we use the Lyapunov function.

Theorem 3.4. *Let* $(\overline{x}, \overline{y})$ *is equilibrium point of the system (3.5). This system is globally asymptotically stable at the said point provided* $x > \overline{x}$.

Proof. Define

$$V(x, y) = \int_{\overline{x}}^{x} \frac{s - \overline{x}}{s} ds + h \int_{\overline{y}}^{y} \frac{t - \overline{y}}{t} dt.$$

By given differential with respect to variable t we have

$$\frac{dV}{dt} = -\beta(x + \overline{x})(x - \overline{x})^2 + (-b + hd)[(x - \overline{x})(y - \overline{y})] - \frac{h}{v}(y - \overline{y})(\frac{y}{x} - \frac{\overline{y}}{\overline{x}}).$$

The system (3.5) is globally asymptotically stable provided the last derivative is negative. As regarding h is arbitrary, we assume that $h = \frac{b}{d}$, and by a calculating we obtain

$$\frac{dV}{dt} = -\beta(x+\overline{x})(x-\overline{x})^2 - \frac{b}{dvx}(y-\overline{y})^2 - \frac{b\overline{y}}{dv}(\frac{1}{x} - \frac{1}{\overline{x}}) = -\beta(x+\overline{x})(x - \overline{x})^2 - \frac{b(y - \overline{y})^2}{dvx} - \frac{b\overline{y}(x - \overline{x})}{dvx\overline{x}},$$

which is negative provided $x > \overline{x}$. Therefore, the proof is completed. □

4. Conclusion

In the present paper we analyzed the effect of different harvest rates on Lotka-Volterra predator-prey model in order to hesitate effect of harvest rate, as a realistic parameter, on this model. In this models, we proved that harvest rate has a positive effect on quick global stability of Lotka-Volterra predator-prey model. The reason of this effecting can be stated by balancing of two prey and predator species. This conclusion can be used in explanation of positive efficiency of controlled use of nature.

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